Arithmetical Functions of Elementary Theorems on The Distribution of Primes

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2024.7.1

Abstract: This is a short note for analytical number theory, introducing some significant arithmetical functions which we use the methodology in mathematical analysis to get a better understanding of the prime distribution on large scale, say when the prime is approaching infinite. This note can used as a brief description for a number theory beginners, or also be used as a short review or check list for student in analytical number theory.

References: This paper is a knowledge combing of «A Friendly Introduction to Number Theory» by J.H. Silverman and «Introduction to Analytical Number Theory» by T.M. Apostol.

1 Introduction

To study the distribution of prime numbers on a large scale, we need to know several important arithmetical functions, and understand the behavior of these functions about prime numbers through analytical methods. The arithmetical functions are deinfed as A real- or complex-valuued funtions which domain is the positive integers, which $f : \mathbb{N} \to \mathbb{C}$, we also call such function as number-theoretic funtion.

2 The Mobius's $\mu(n)$

Definition 2.1 *The Mobius's funtion μ defined as:*

$$\mu(1) = 1$$

if $n > 1$, by unique factorization theory we have $n = p_1^{a_1} p_2^{a_2} ... p_k^{a_k}$.

$$\mu(n) = \begin{cases} (-1)^k, & \textit{if } a_i = 1, \forall i \\ 0, & \textit{otherwise} \end{cases}$$

Theorem 2.1 $\forall n \in \mathbb{N}$ *we have*

$$\sum_{d|n} \mu(d) = [1/n] = \begin{cases} 1, & \textit{if } n = 1 \\ 0, & \textit{if } n > 1. \end{cases}$$

Proof

$$\begin{aligned} \sum_{d|n} \mu(d) &= \mu(1) + \mu(p_1) + ... + \mu(p_k) + \mu(p_1 p_2) + ... + \mu(p_{k-1} p_k) + ... + \mu(p_1 p_2 ... p_k) \\ &= 1 + \begin{pmatrix} k \\ 1 \end{pmatrix} (-1) + \begin{pmatrix} k \\ 2 \end{pmatrix} (-1)^2 + ... + \begin{pmatrix} k \\ k \end{pmatrix} (-1)^k = (1-1)^k = 0 \end{aligned}$$

□

3 The Euler's totient $\varphi(n)$

Definition 3.1

$$\varphi(n) = |\{(a, n) = 1, 1 \leq a \leq n\}|$$

Euler totient function is defined to be the number of positive integers not exceeding n which are relatively prime to n.

Theorem 3.1 $\forall n \geq 1$, *we have*

$$\sum_{d|n} \varphi(d) = n$$

Proof We define $A(d) := \{k : (k, n) = d, 1 \leq k \leq n\}$, notice that if $f(d) := |A(d)|$, then $\sum_{d|n} f(d) = n$. We know $(k, n) = d$, if and only if $(k/d, n/d) = 1$. Which is easy to find $f(d) = \varphi(n/d)$, hence we have $\sum_{d|n} \varphi(n/d) = n$, which is equivalent to the statement $\sum_{d|n} \varphi(n) = n$. □

Theorem 3.2 $\forall n \geq 1$, *we have*

$$\varphi(n) = \sum_{d|n} \mu(d)\frac{n}{d}.$$

Proof

$$\begin{aligned}\varphi(n) &= \sum_{k=1}^{n}[\frac{1}{(n,k)}] \\ &= \sum_{k=1}^{n}\sum_{d|(n,k)} \mu(d) \\ &= \sum_{k=1}^{n}\sum_{d|n,d|k} \mu(d)\end{aligned}$$

for fixed $d|n$, we sum over $k \leq n$ in the form $k = qd$ for some q with 1≤ q ≤ n/d, we have

$$\varphi(n) = \sum_{d|n}\sum_{q=1}^{n/d} \mu(d) = \sum_{d|n} \mu(d) \sum_{q=1}^{n/d} 1 = \sum_{d|n} \mu(d)\frac{n}{d}.$$ □

Proposition 3.1 *Euler's totient satisfies:*
(a) $\varphi(p^a) = p^a - p^{a-1}$ *for p a prime and a≥1.*
(b) $\varphi(xy) = \varphi(x)\varphi(y)(z/\varphi(z))$, *where* $z = (x, y)$.
(c) *if* $a|b$, *then* $\varphi(a)|\varphi(b)$.
(d) $2|\varphi(n)$, $\forall n \geq 3$. *Moreover, if n has r disinct odd prime factors, then* $2^r|\varphi(n)$

Proof We give a proof one by one
(a) $\varphi(p^a) = p^a - |\{(a, n) \neq 1, 1 \leq a \leq n\}| = p^a - |\{p, 2p, ...p^{a-1}p\}| = p^a - p^{a-1}$.
(b) By unique factorization, $n = p_1^{a_1}p_2^{a_2}...p_k^{a_k}$. Consider the product

$$\prod_{p|n}(1 - \frac{1}{p}) = \prod_{i=1}^{k}(1 - \frac{1}{p_i})$$

Where $p_1, p_2, ...p_k$ are the distinct prime divisors of n. And by **Theorem 3.2** we have

$$\begin{aligned}\varphi(n) &= \sum_{d|n} \mu(d)\frac{n}{d} = n\sum_{d|n}\frac{\mu(d)}{d} \\ &= n(1 - \sum \frac{1}{p_i} + \sum \frac{1}{p_ip_j} - \sum \frac{1}{p_ip_jp_r} + ... + \frac{(-1)^k}{p_1p_2...p_k}) \\ &= n(\prod_{i=1}^{k}(1 - \frac{1}{p_i}))\end{aligned}$$

The product like $\sum \frac{1}{p_i p_j p_r}$ is represented as all possible producties of disinct prime factors of n taken three at a time, and there should be $\begin{pmatrix} k \\ 3 \end{pmatrix}$ fractions in this sum. The sum is one divisor of n, and the molecule is actually the Mobius's value of the divisor.
Hence we have

$$\frac{\varphi(n)}{n} = \prod_{p|n}(1 - \frac{1}{p})$$

We have the fact that if p is prime, we have $d|xy \Rightarrow d|x \ or \ d|y$, also $p|x \ and \ p|y \Rightarrow p|(x, y)$. Hence

$$\frac{\varphi(xy)}{xy} = \prod_{p|xy}(1 - \frac{1}{p}) = \frac{\prod_{p|x}(1 - \frac{1}{p}) \prod_{p|y}(1 - \frac{1}{p})}{\prod_{p|(x,y)}(1 - \frac{1}{p})} = \frac{\frac{\varphi(x)}{x}\frac{\varphi(y)}{y}}{\frac{\varphi(z)}{z}}$$

(c) $\exists$ c where $1 \leq c \leq b, \ s.t. \quad b = ac$. If $c = b$, then $\varphi(a) = 1$, trivially satisfied
If c<b, from (b) we have

$$\varphi(b) = \varphi(ac) = \varphi(a)\varphi(c)\frac{d}{\varphi(d)}$$

By induction, the result (c) is hold for $b = 1$. Supppose it holds $\forall \mathbb{N} < b$, then $\varphi(d)|\varphi(c)$ since $d|c$, hence we have $\varphi(a)|\varphi(b)$ as required.
(d) if n has no odd prime factors, then n must has a form of 2^a for some $a \geq 1$, and by (a) shows that $\varphi(2^a)$ is even for $a \geq 2$. If n has at least one odd prime factor, by the proof in (b) we have

$$\varphi(n) = n \prod_{p|n} \frac{p-1}{p} = \frac{n}{\prod_{p|n} p} \prod_{p|n}(p - 1)$$

Notice that $\frac{n}{\prod_{p|n} p}$ is a integer, and $\prod_{p|n}(p - 1)$ is even with an even factor$(p_i - 1)$, where p_i is the only one odd factor of n, hence $\varphi(n)$ is even. Moreover, one odd prime factor contributes a factor 2 to the product, so we proved $2^r|\varphi(n)$ if n has r disinct odd prime factors as required. □

4 The Mangoldt's $\Lambda(n)$

Definition 4.1 $\forall n \in \mathbb{Z}, n \geq 1$

$$\Lambda(n) = \begin{cases} log \ p, & if \ \exists m \geq 1, s.t. \quad n = p^m \\ 0, & otherwise \end{cases}$$

where p is a prime.

Theorem 4.1 $\forall n \geq 1$, *we have*

$$log \ n = \sum_{d|n} \Lambda(d)$$

Proof For n = 1, the proof is trivial. Now consider n > 1. By unique factorization we have

$$n = \prod_{k=1}^{r} p_k^{a_k}$$

Hence

$$log \ n = \sum_{k=1}^{r} a_k \ log \ p_k$$

Recall the sum $\sum_{d|n} \Lambda(n)$, we only count nonzero terms, we have

$$\sum_{d|n} \Lambda(n) = \sum_{k=1}^{r} \sum_{m=1}^{a_k} \Lambda(p_k^m) = \sum_{k=1}^{r} \sum_{m=1}^{a_k} log\ p_k = \sum_{k=1}^{r} a_k log\ p_k = log\ n$$

In Mangoldt's, all the number smaller than $n = p_1^{a_1} p_2^{a_2} ... p_k^{a_k}$, only those with the form p_k^m have nonzero value, where m runs over from 1 to a_k, k runs over from 1 to r. $\square$

Lemma 4.1 *Mobius inversion formula*

$$f(n) = \sum_{d|n} g(d)$$

implies

$$g(n) = \sum_{d|n} f(d)\mu(\frac{n}{d})$$

*Recall **Theorem 3.1** and **Theorem 3.2**.*

$$n = \sum_{d|n} \varphi(d) \ and\ \varphi(n) = \sum_{d|n} d\mu(\frac{n}{d})$$

Theorem 4.2 $\forall n \geq 1$, *we have*

$$\Lambda(n) = \sum_{d|n} \mu(d) log \frac{n}{d} = -\sum_{d|n} \mu(d) log\ d$$

Proof By Mobius inversion formula

$$\Lambda(n) = \sum_{d|n} \mu(d) log \frac{n}{d} = log\ n \sum_{d|n} \mu(d) - \sum_{d|n} \mu(d) log\ d$$ $\square$

where $log\ n \sum_{d|n} \mu(d) = 0$.

5 Liouville's $\lambda(n)$

Definition 5.1 *We define* $\lambda(1) = 1$, *and*

$$\lambda(\prod_{i=1}^{k} p_i^{a_i}) = (-1)^{\sum_{i=1}^{k} a_i}$$

by unique factorization $n = p_1^{a_1} p_2^{a_2} ... p_k^{a_k}$ *for some primes* p_i *and* $a_1 \geq 1$, *then* $\lambda(n) = (-1)^{a_1 + a_2 + ... + a_k}$.

Theorem 5.1 $\forall n \geq 1$, *we have*

$$\sum_{d|n} \lambda(d) = \begin{cases} 1, & \textit{if } n \textit{ is a square} \\ 0, & \textit{otherwise} \end{cases}$$

Also, $\lambda^{-1}(n) = |\mu(n)|$ *for all* n, *where* $\mu(n)$ *is Mobius's.*

Proof Let $g(n) := \sum_{d|n} \lambda(d)$, consider

$$g(p^a) = \sum_{d|p^a} \lambda(d) = 1 + \lambda(p) + \lambda(p^2) + ... + \lambda(p^a)$$

$$= 1 - 1 + 1 - ... + (-1)^a = \begin{cases} 1, & \text{if } 2 \mid n \\ 0, & \text{if } 2 \nmid n \end{cases}$$

So if $n = \prod_{i=1}^{k} p_i^{a_i}$, by the multiplicativity of $\lambda(n)$, $g(n)$ is also multiplicative and we have $g(n) = \prod_{i=1}^{k} g(p_i^{a_i})$, to make $g(n) = 1$, all the exponents a_i must be even, if there is one odd and the product will be 0. This shows that $g(n) = 1$ if n is square, and $g(n) = 0$ otherwise. Also, $\lambda^{-1}(n) = \mu(n)\lambda(n) = \mu^2(n) = |\mu(n)|$. $\square$

6 The divisor function $\sigma_\alpha(n)$

Definition 6.1 *For $\alpha \in \mathbb{R}$ or $\mathbb{C}$, and $\forall n \geq 1, n \in \mathbb{N}$, we define $\sigma_\alpha(n)$ as the sum of αth powers of the divisors of n.*

$$\sigma_\alpha(n) = \sum_{d|n} d^\alpha$$

*The **number** of divisors of n denoted by*

$$d(n) := \sigma_0(n)$$

*The **sum** of divisors of n denoted by*

$$\sigma(n) := \sigma_1(n)$$

Since σ_α is multiplicative we have

$$\sigma_\alpha(\prod_{i=1}^{k} p_i^{a_1}) := \prod_{i=1}^{k} \sigma_\alpha(p_i^{a_i})$$

To compute a single term like $\sigma_\alpha(p^a)$, we know the divisors of prime power p^a are $1, p, p^2, ..., p^a$, hence

$$\begin{aligned}\sigma_\alpha(p^a) &= 1^\alpha + p^\alpha + p^2\alpha + ... + p^a\alpha \\ &= \begin{cases} \frac{p^\alpha(a+1)-1}{p^\alpha-1}, & \text{if } \alpha \neq 0 \\ a+1, & \text{if } \alpha = 0 \end{cases}\end{aligned}$$

Theorem 6.1 *For $n \geq 1$, we have*

$$\sigma_\alpha^{-1}(n) = \sum_{d|n} d^\alpha \mu(d)\mu(\frac{n}{d})$$

7 Prime counting function $\pi(x)$

Consider $\pi(x)$, as the number of primes not exceeding x for some x > 0. By Euclid's proof by contridiction, the number of primes has no upper bound, thus $\pi(x) \to \infty$ as $x \to \infty$. The equation describe behavior of $\pi(x)$ as x goes large is the **Prime Number Theorem**.

Theorem 7.1

$$\lim_{x\to\infty} \frac{\pi(x) log\ x}{x} = 1$$

In which, an equivalent way

$$\pi(x) \sim \frac{x}{log\ x} \text{ as } x \to \infty$$

We say that $\pi(x)$ is asymptotic to $x/log\ x$. Also, the prime number theorem is equivalent to the asymptotic formula

$$\sum_{n\leq x} \Lambda(n) \sim x \text{ as } x \to \infty$$

The Prime Number Theorem will be introduced in section 10

8 Chebyshev's $\psi(x)$ and $\vartheta(x)$

Definition 8.1 *For $x > 0$, we define*

$$\psi(x) = \sum_{n \le x} \Lambda(n)$$

Recall Mangoldt's $\Lambda(n)$ only counts when n is in the form of p^m where p is prime and $m \ge 1$. So in Chebyshev's $\psi(x)$, the function value is nonzero only if there is some $\mathbb{N}$ s.t. $p^m \le x$ given. Which is required that $p \le x^{1/m}$. When m runs from 1 to ∞ in the sum, the sum on p will be zero since $x^{1/2} < 2$, which is $(1/m)log\ x < log\ 2$, or if $m > log_2\ x$.
Therefore we now have

$$\psi(x) = \sum_{m \le log_2 x} \sum_{p \le x^{1/m}} log\ p$$

Definition 8.2 *For $x > 0$, we define*

$$\vartheta(x) = \sum_{p \le x} log\ p$$

where p runs over all prime $\le x$, and it is clear that we have

$$\psi(x) = \sum_{m \le log_2 x} \vartheta(x^{1/m})$$

Theorem 8.1 *For $x > 0$, we have*

$$0 \le \frac{\psi(x)}{x} - \frac{\vartheta(x)}{x} \le \frac{(log\ x)^2}{2\sqrt{x} log\ 2}$$

Proof From $\psi(x) = \sum_{m \le log_2 x} \vartheta(x^{1/m})$ we have

$$0 \le \psi(x) - \vartheta(x) = \sum_{2 \le m \le log_2 x} \vartheta(x^{1/m})$$

Recall the upper bound of Chebyshev's $\vartheta(x)$

$$\vartheta(x) \le \sum_{p \le x} log\ x \le xlog\ x$$

First inequality is because $log\ p \le \log x\ \forall p \le x$, second is trivial that $p \le x$, so

$$0 \le \psi(x) - \vartheta(x) \le \sum_{2 \le m \le log_2 x} x^{1/m} log(x^{1/m}) \le (log_2 x)\sqrt{x} log\sqrt{x} = \frac{log\ x}{log\ 2}\frac{\sqrt{x}}{2} log\ x = \frac{(log\ x)^2}{2\sqrt{x} log\ 2}$$ □

Theorem 8.2

$$\lim_{x \to \infty} (\frac{\psi(x)}{x} - \frac{\vartheta(x)}{x}) = 0$$

Proof This is trivial by **Theorem 8.1** and apply Squeeze Theorem. □

9 Abel's identity A(x)

Definition 9.1 *For any arithmetical function $a(x)$, we defined*

$$A(x) = \sum_{n \le x} a(n)$$

$A(x) = 0,\ \forall x < 1$

Theorem 9.1 *Assume $f : [y, x] \to \mathbb{C}$ is differentiable, we have*

$$\sum_{y<n\leq x} a(n)f(n) = A(x)f(x) - A(y)f(y) - \int_x^y A(t)f'(t)dt$$

Proof Notice all arithmetical function defined on $\mathbb{N}$, so first let $k := [x]$ and $m := [y]$. So $A(x) = A(k)$ and $A(y) = A(m)$, and by the definition of $A(x)$ we have

$$\sum_{y<n\leq x} a(n)f(n) = \sum_{n=m+1}^{k} a(n)f(n) = \sum_{n=m+1}^{k} \{A(n) - A(n-1)\}f(n)$$

By distributive, and with the fact that $\sum_{n=m+1}^{k} A(n-1)f(n) = \sum_{n=m}^{k-1} A(n)f(n+1)$, we have

$$\sum_{n=m+1}^{k} A(n)f(n) - \sum_{n=m}^{k-1} A(n)f(n+1) = \sum_{n=m+1}^{k-1} A(n)\{f(n) - f(n+1)\} + A(k)f(k) - A(m)f(m+1)$$

notice that $f(n) - f(n+1)$ can be replaced by $-\int_n^{n+1} f'(t)dt$, therefore

$$\begin{aligned}
\sum_{y<x\leq x} a(n)f(n) &= -\sum_{n=m+1}^{k-1} A(n)\int_n^{n+1} f'(t)dt + A(k)f(k) - A(m)f(m+1) \\
&= -\sum_{n=m+1}^{k-1} \int_n^{n+1} A(t)f'(t)dt + A(k)f(k) - A(m)f(m+1) \\
&= -\int_{m+1}^{k} A(t)f'(t)dt + A(x)f(x) - \int_k^x A(t)f'(t)dt - A(y)f(y) - \int_y^{m+1} A(t)f'(t)dt \\
&= A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)dt
\end{aligned}$$

In the penultimate line, we use the fact that $A(k)f(k) = A(k)f(k) - A(x)f(x) + A(x)f(x) = A(x)f(x) - \int_k^x A(t)f'(t)dt$, and the same as $A(m)f(m+1)$. □

Proof The Proof for Abel's identity can also be done by **Riemann-Stieltjes integration**, consider A(x) is jump a(n) at each integer n, the sum can be written as Riemann-Stieltjes integral

$$\begin{aligned}
\sum_{y<n\leq x} a(n)f(n) &= \int_y^x f(t)dA(t) \\
&= f(x)A(x) - f(y)A(y) - \int_y^x A(t)df(t) \\
&= f(x)A(x) - f(y)A(y) - \int_y^x A(t)f'(t)dt
\end{aligned}$$

□

Theorem 9.2 *For $x \geq 2$, we have*

$$\vartheta(x) = \pi(x)log\ x - \int_x^2 \frac{\pi(t)}{t}dt$$

$$\pi(x) = \frac{\vartheta(x)}{log\ x} + \int_2^x \frac{\vartheta(x)}{t\ log^2 t}dt$$

Proof Let a(n) denote the counting function of primes

$$a(n) = \begin{cases} 1, & \text{if } if\ n\ is\ prime \\ 0, & \text{otherwise .} \end{cases}$$

By the definition, is clear to see

$$\vartheta(x) = \sum_{p\le x} log\ p = \sum_{1<n\le x} a(n) log\ n$$
$$\pi(x) = \sum_{p\le x} 1 = \sum_{1<n\le x} a(n)$$

Recall Abel's identity and take $f(x) = log\ x$

$$\sum_{y<n\le x} a(n)f(n) = A(x)f(x) - A(y)f(y) - \int_x^y A(t)f'(t)dt$$

$$\begin{aligned}\vartheta(x) = \sum_{1<n\le x} a(n)log\ n &= \pi(x)log\ x - \pi(1)log\ 1 - \int_1^x \frac{\pi(t)}{t}dt \\ &= \pi(x)log\ x - \pi(1)log\ 1 - \int_1^2 \frac{\pi(t)}{t}dt - \int_2^x \frac{\pi(t)}{t}dt \\ &= \pi(x)log\ x - \int_2^x \frac{\pi(t)}{t}dt\end{aligned}$$

where $\pi(t) = 0,\ \forall t < 2$. Then we define $b(n) := a(n)log\ n$, again by the definition

$$\vartheta(x) = \sum_{n\le x} b(n)$$
$$\pi(x) = \sum_{3/2<n\le x} b(n)\frac{1}{log\ n}$$

Here we sum n from 3/2 not from 1 in $\pi(x)$, is seems nothing change with previous value, but in Abel's identity, this summation interval avoids infinity where $'1/\log 1'$not exists. We take $f(x) = 1/log\ x$ and $y = 3/2$ in Abel's identity and we get

$$\begin{aligned}\pi(x) &= \frac{\vartheta(x)}{log\ x} - \frac{\vartheta(3/2)}{log 3/2} + \int_{3/2}^x \frac{\vartheta(t)}{tlog^2\ t}dt \\ &= \frac{\vartheta(x)}{log\ x} - \frac{\vartheta(3/2)}{log 3/2} + \int_{3/2}^2 \frac{\vartheta(t)}{tlog^2\ t}dt + \int_2^x \frac{\vartheta(t)}{tlog^2\ t}dt \\ &= \frac{\vartheta(x)}{log\ x} + \int_2^x \frac{\vartheta(t)}{tlog^2\ t}dt\end{aligned}$$

□

10 Prime Number Theorem

Theorem 10.1 *The following relation are logically equivalent*

$$\lim_{x\to\infty} \frac{\pi(x)log\ x}{x} = 1$$
$$\lim_{x\to\infty} \frac{\vartheta(x)}{x} = \lim_{x\to\infty} \frac{\sum_{p\le x} log\ p}{x} = 1$$
$$\lim_{x\to\infty} \frac{\psi(x)}{x} = \lim_{x\to\infty} \frac{\sum_{n\le x} \Lambda(x)}{x} = 1$$

Proof For $x \ge 2$, by **Theorem 9.2** we have

$$\frac{\vartheta(x)}{x} = \frac{\pi(x)log\ x}{x} - \frac{1}{x}\int_2^x \frac{\pi(t)}{t}dt$$
$$\frac{\pi(x)log\ x}{x} = \frac{\vartheta(x)}{x} + \frac{log\ x}{x}\int_2^x \frac{\vartheta(t)dt}{t\ log^2 t}$$

For $\lim_{x\to\infty}\frac{\pi(x)log\ x}{x}=1$, implies $\frac{\pi(t)}{t}=O(\frac{1}{log\ t}), \forall t\geq 2$, so we have

$$\frac{1}{x}\int_2^x\frac{\pi(t)}{t}dt=O(\frac{1}{x}\int_2^x\frac{dt}{log\ t})$$

Now consider the upper bound

$$\int_2^x\frac{dt}{log\ t}=\int_2^{\sqrt{x}}\frac{dt}{log\ t}+\int_{\sqrt{x}}^x\frac{dt}{log\ t}\leq\frac{\sqrt{x}}{log2}+\frac{x-\sqrt{x}}{log\sqrt{x}}$$

Hence

$$\lim_{x\to\infty}\frac{1}{x}\int_2^x\frac{dt}{log\ t}=0$$

which we have

$$\lim_{x\to\infty}\frac{\vartheta(x)}{x}=\lim_{x\to\infty}(\frac{\pi(x)log\ x}{x}-\frac{1}{x}\int_2^x\frac{\pi(t)}{t}dt)=\lim_{x\to\infty}\frac{\pi(x)log\ x}{x}$$

To show the other side, we can follow the same step, from $\lim_{x\to\infty}\frac{\vartheta(x)}{x}=1$, we have $\vartheta(t)=O(t)$, so

$$\frac{log\ x}{x}\int_2^x\frac{\vartheta(t)d(t)}{t\ log^2t}=O(\frac{log\ x}{x})\int_2^x\frac{dt}{log^2t}$$

Now consider the upper bound

$$\int_2^x\frac{dt}{log^2t}=\int_2^{\sqrt{x}}\frac{dt}{log^2\ t}+\int_{\sqrt{x}}^x\frac{dt}{log^2\ t}\leq\frac{\sqrt{x}}{log^2\ 2}+\frac{x-\sqrt{x}}{log^2\sqrt{x}}$$

Hence

$$\lim_{x\to\infty}\frac{log\ x}{x}\int_2^x\frac{dt}{log^2\ t}=0$$

which we have

$$\lim_{x\to\infty}\frac{\pi(x)log\ x}{x}=\lim_{x\to\infty}(\frac{\vartheta(x)}{x}+\frac{log\ x}{x}\int_2^x\frac{\vartheta(t)dt}{t\ log^2t})=\lim_{x\to\infty}\frac{\vartheta(x)}{x}$$

By Theorem **Theorem 10.2**, we have proved that $\lim_{x\to\infty}\frac{\vartheta(x)}{x}=1$ and $\lim_{x\to\infty}\frac{\psi(x)}{x}=1$ are equivalent. □

Theorem 10.2 *Let p_n denote the nth prime. The following asymptotic relation are equivalent*

$$\lim_{x\to\infty}\frac{\pi(x)log\ x}{x}=1\quad(1)$$
$$\lim_{x\to\infty}\frac{\pi(x)log\pi(x)}{x}=1\quad(2)$$
$$\lim_{n\to\infty}\frac{p_n}{nlog\ n}=1\quad(3)$$

Proof We give the proof by showing (1) ⇒ (2) ⇒ (3) ⇒ (2) ⇒ (1)
Asuume (1) holds, we have

$$\lim_{x\to\infty}log(\frac{\pi(x)log\ x}{x})=\lim_{x\to\infty}[log\pi(x)+loglog\ x-log\ x]=log1=0$$

which

$$\lim_{x\to\infty}[log\ x(\frac{log\pi(x)}{log\ x}+\frac{loglog\ x}{log\ x}-1)]=0$$

We know $\lim_{x\to\infty} log\ x = \infty$, it must follows that

$$\lim_{x\to\infty}[(\frac{log\pi(x)}{log\ x} + \frac{loglog\ x}{log\ x} - 1)] = 0$$
$$\lim_{x\to\infty}\frac{log\pi(x)}{log\ x} = 1$$

together with (1) gives us (2). Now we asuume (2) holds, we know by definition $\pi(p_n) = n$, let $x := p_n$ then we have

$$\lim_{x\to\infty}\frac{\pi(x)log\pi(x)}{x} = \lim_{n\to\infty}\frac{n\ log\ n}{p_n} = 1$$

which is trivial, we now prove the other side, assume (3) holds. For fixed x given, we define n by letting p_n and p_{n+1} be the closest two primes to x, that is

$$p_n \le x \le p_{n+1}$$

such n is unique and we have $u = \pi(x)$, consider the upper bound of a fixed n in (3)

$$\frac{p_n}{n\ log\ n} \le \frac{x}{n\ log\ n} < \frac{p_{n+1}}{n\ log\ n} = \frac{p_{n+1}}{(n+1)log\ (n+1)}\frac{(n+1)log\ (n+1)}{n\ log\ n}$$

Taking $x\to\infty$ and apply squeeze theorem we have

$$\lim_{n\to\infty}\frac{x}{n\ log\ n} = \lim_{x\to\infty}\frac{x}{\pi(x)log\pi(x)} = 1$$

Finally we show (2) $\Rightarrow$ (1), from (2) we have

$$\lim_{n\to\infty} log(\frac{\pi(x)log\pi(x)}{x}) = \lim_{x\to\infty}(log\pi(x) + loglog\pi(x) - log\ x) = log1 = 0$$

which

$$\lim_{x\to\infty}[log\pi(x)(1 + \frac{loglog\pi(x)}{log\pi(x)} - \frac{log\ x}{log\pi(x)})] = 0$$

We know $\lim_{x\to\infty} log\ x = \infty$, it must follows that

$$\lim_{x\to\infty}(1 + \frac{loglog\pi(x)}{log\pi(x)} - \frac{log\ x}{log\pi(x)}) = 0$$
$$\lim_{x\to\infty}\frac{log\ x}{log\pi(x)} = 1$$

together with (1) gives us (2) $\square$

Lemma 10.1 *Legendre's identity. For every $x \ge 1$ we define*

$$\alpha(p) := \sum_{m=1}^{\infty}[\frac{x}{p^m}]$$

then we have

$$[x]! = \prod_{p\le x}\alpha(p)$$

where p is prime and the product is extended over all primes $\le x$, and it is clear that the sum for $\alpha(p)$ is finite since $[x/p^m] = 0, \forall p > x$.

Proof Recall Mangoldt's $\lambda(n) = 0$ is nonzero only if n is in the form of p^m for some prime p and $m \ge 1$ and $\Lambda(p^m) = log\ p$, so we have

$$log[x]! = \sum_{n\le x}\lambda(n)[\frac{x}{n}] = \sum_{p\le x}\sum_{m=1}^{\infty}[\frac{x}{p^m}]log\ p = \sum_{p\le x}\alpha(p)log\ p$$ $\square$

Theorem 10.3 *Consider the upper bound and lower bound for fixed $n \geq 2, n \in \mathbb{N}$, we have*

$$\frac{1}{6}\frac{n}{log\ n} < \pi(n) < 6\frac{n}{log\ n}$$

Proof Notice the inequality

$$2^n \leq \frac{(2n)!}{n!n!} = \begin{pmatrix} 2n \\ n \end{pmatrix} < \sum_{k=0}^{2n} \begin{pmatrix} 2n \\ k \end{pmatrix} = (1+1)^{2n} = 4^n$$

which is

$$n\ log\ 2 \leq log(2n)! - 2\ log\ n! < n\ log\ 4$$

By **Lemma 10.1**, we deinfed

$$\alpha(p) = \sum_{m=1}^{[\frac{log\ n}{log\ p}]} [\frac{n}{p^m}]$$

and we have

$$log\ n! = \sum_{p \leq n} \alpha(p) log\ p$$

Hence

$$\begin{aligned} n\ log\ 2 \leq log(2n)! - 2log\ n! &= \sum_{p \leq 2n} \sum_{m=1}^{[\frac{log\ 2n}{log\ p}]} \{[\frac{2n}{p^m}] - 2[\frac{n}{p^m}]\} log\ p \qquad (1) \\ &\leq \sum_{p \leq 2n} (\sum_{m=1}^{[\frac{log\ 2n}{log\ p}]} 1)\ log\ p \\ &\leq \sum_{p \leq 2n} log\ 2n \\ &= \pi(2n) log\ 2n \end{aligned}$$

which gives us

$$\begin{aligned} \pi(2n) \geq & \frac{n\ log\ 2}{log\ 2n} = \frac{2n}{log\ 2n}\frac{log\ 2}{2} > \frac{2n}{log\ 2n}\frac{1/2}{2} = \frac{1}{4}\frac{2n}{log\ 2n} \\ \pi(2n+1) \geq & \pi(2n) > \frac{1}{4}\frac{2n}{log\ 2n} > \frac{1}{4}\frac{2n}{2n+1}\frac{2n+1}{log(2n+1)} \geq \frac{1}{4} \cdot \frac{2}{3}\frac{2n+1}{log(2n+1)} = \frac{1}{6}\frac{2n+1}{log(2n+1)} \end{aligned}$$

That is one side of the inequality

$$\pi(n) > \frac{1}{6}\frac{n}{log\ n}$$

To show the other side, recall (1) and extract the term to $m = 1$

$$\begin{aligned} log(2n)! - 2log\ n! &= \sum_{p \leq 2n} \sum_{m=1}^{[\frac{log\ 2n}{log\ p}]} \{[\frac{2n}{p^m}] - 2[\frac{n}{p^m}]\} log\ p \\ &\geq \sum_{p \leq 2n} \{[\frac{2n}{p^1}] - 2[\frac{n}{p^1}]\} log\ p \\ &\geq \sum_{n < p \leq 2n} \{[\frac{2n}{p}] - 2[\frac{n}{p}]\} log\ p \\ &= \sum_{n < p \leq 2n} log\ p \\ &= \vartheta(2n) - \vartheta(n) \end{aligned}$$

Recall Chebyshev's $\vartheta(x)$, and the inequality at the beginning of the proof $n\ log\ 2 \leq log(2n)! - 2\ log\ n! < n\ log\ 4$ gives us

$$\begin{aligned}
\vartheta(2n) - \vartheta(n) &< n\ log\ 4 \\
\vartheta(2^{r+1}) - \vartheta(2^r) &< 2^r log\ 4 = 2^{r+1} log\ 2 \\
\vartheta(2^{k+1}) = \vartheta(2^{k+1}) - \vartheta(1) &= [\vartheta(2^{k+1}) - \vartheta(2^k)] + [\vartheta(2^k) - \vartheta(2^{k-1})] + ... + [\vartheta(2^2) - \vartheta(2)] \\
&< 2^{k+1} log\ 2 + 2^k log\ 2 + ... + 2^2 log\ 2 \\
&< 2^{k+1} log\ 2 + 2^k log\ 2 + ... + 2^2 log\ 2 + 2 log\ 2 \\
&= 2^{k+2} log\ 2
\end{aligned}$$

Now we fixed unique $k, s.t.\ 2^k \leq n < 2^{k+1}$ and we obtain

$$\vartheta(n) \leq \vartheta(2^{k+1}) < 2^{k+2} log\ 2 \leq 4n\ log\ 2$$

Consider $0 < \alpha < 1$ we have

$$\begin{aligned}
(\pi(n) - \pi(n^\alpha)) log\ n^\alpha &< \sum_{n^\alpha < p \leq n} log\ p \leq \vartheta(n) < 4n\ log\ 2 \\
\pi(n) &< \frac{4n\ log\ 2}{\alpha\ log\ n} + \pi(n^\alpha) \\
&< \frac{4n\ log\ 2}{\alpha log\ n} + n^\alpha \\
&= \frac{n}{log\ n}(\frac{4\ log\ 2}{\alpha} + \frac{log\ n}{n^{1-\alpha}})
\end{aligned}$$

With the fact that $\forall c > 0,\ \forall x \geq 1,\ n^{-c} log\ n \leq 1/(ce), \forall n \geq 1$. Taking $\alpha = 2/3$ we find

$$\pi(n) < \frac{n}{log\ n}(6\ log\ 2 + \frac{3}{e}) < 6\frac{n}{log\ n}$$

Which is the other side of the inequality. $\square$

Theorem 10.4 *For $n \geq 1$ with p_n denotes the nth prime, we have*

$$\frac{1}{6} n\ log\ n < p_n < 12(n\ log\ n + n\ log\ \frac{12}{e})$$

Proof Let $k := p_n$, then by definition $k \geq 2$ and $n = \pi(k)$, from **Theorem 10.3** we have

$$n = \pi(k) < 6\frac{k}{log\ k} = 6\frac{p_n}{log\ p_n}$$

So

$$p_n > \frac{1}{6} n\ log\ p_n > \frac{1}{6} n\ log\ n$$

which is the lower bound of the inequality, to show the other side, again with **Theorem 10.3**

$$n = \pi(k) > \frac{1}{6}\frac{k}{log\ k} = \frac{1}{6}\frac{p_n}{log\ p_n}$$

So

$$\begin{aligned}
p_n &< 6n\ log\ p_n \\
\sqrt{p_n} &< \frac{12}{e} n
\end{aligned}$$

with the fact that $log\ x \leq (2/e)\sqrt{x},\ \forall x \geq 1$. Hence

$$p_n^{1/2} = n\frac{\sqrt{p_n}}{n} < n\frac{12}{e}$$
$$\frac{1}{2}\ log\ p_n < log\ n + log\ \frac{12}{e}$$

Finally, with $p_n < 6n\ log\ p_n$, gives us

$$p_n < 6n(2\ log\ n + 2\ log\frac{12}{e})$$

which is the upper bound of the inequality. $\square$

Waiting for replenishment...