Futurama Theorm: Group and permutations in body changing problem

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Abstract: In the Futurama episode <The Prisoner of Benda>, Professor Farnsworth and Amy build a machine that allows two people to switch bodies. It quickly becomes clear that after two bodies have been swapped, the machine cannot swap the sane two bodies back again, ever. So can everyone get their own body back by adding more people? We will show the condition that allow this happen and also provide the way to do so.

Keywords: Permutation, Transposition, Group theory

1 Futurama Theorem

Let S be a finite set, and let x and y be distinct objects that do not belong to S. Any permutation of S can be reduced to the identity permutation by applying a sequence of distinct transpositions of $S \cup \{x, y\}$, each of which includes at least one of x, y. In mathematical words:

$$\begin{aligned} &For\ a\ finite\ set\ S = \{a_1, a_2, a_3, ...a_n\},\ \forall\ \sigma \in S_n,\ \exists\ m_i, n_i \in S,\ s.t. \\ &\sigma^{-1} = (xm_1)(xm_2)...(xm_j)(yn_1)(yn_2)...(yn_k),\ where\ x, y \notin S. \end{aligned}$$

2 You should know before you start

This is the proof sheet of <Permutations: The Prisoner of Benda> by Adam-Christiaan Van Roosmalen. You should first read Adam's paper and think about the question in assignments as if it is possible that everyone can switch back when there are three and four people. Find couple of students and play the game if you can. Then have a little look at this paper to get some hints and try as mush as you can to obtain the proof.

3 Permutations and Transpositions

Definition 3.1 *The symmetry group S_n is defined on a set $S := \{1, 2, 3, ..., n\}$, s.t.*

$$S_n := \{All\ bijective\ mapping\ from\ S\ to\ S\}$$

equipped the composition of mapping as the operator.

Example 3.1 *Under this definition, it is easy to observe that there are a cycle structure hide in permutations. For S_2 we only have two elements, that is, $Id : 1 \mapsto 1, 2 \mapsto 2$, or $\sigma : 1 \mapsto 2, 2 \mapsto 1$. Consider σ, it is a permutation of two items, and can be denoted as (12). That is to say σ sends 1 to 2 and 2 to 1. Moreover $\sigma^2 = (12)^2 = Id$, we say σ is a element of order 2, and there is a* ***cycle*** *of 1 and 2.*

Definition 3.2 *A Transposition is a permutation between two elements, and we use (ab) to denote the permutation* $f : a \mapsto b, b \mapsto a$.

Proposition 3.1 *Every cycle in* S_n *is a product of transpositions.*

Example 3.2 *Condier* S_3*, there is a permutation* $\phi : 1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 1$*. It is easy to verify* ϕ *is a cycle of 1, 2, 3, denoted as (123). Moreover (123) = (23)(13) = (13)(12), as you can see, the transposition representation of a determined cycle is not unique.*

The key to solve Futurama problem is that everyone bijective mapping always generate cycle structure for domain set (why? Try to show this by contradiction). If a man's mind is so-called the man himself, then the body swaps is just given by a bijective function $f : \textit{people} \to \textit{people}$, we say that

$$\begin{aligned} f(p) = q &\iff person\ q's\ mind\ is\ in\ body\ p \\ &\iff person\ p\ holds\ the\ name\ tag\ of\ person\ q \end{aligned}$$

For a body switching of three people, each time only two people go into the machine and switch their bodies, so each time is a transposition of people. We consider if we take two rounds of switching, say (13)(12). We first change 1 and 2's bodies, then change 2 and 3's body. This is equivalent to we put 1's mind in 2's body, 2's mind in 3's body, 3's mind in 1's body just in one round, i.e. (13)(12) = (123).

$$\begin{array}{ccccccccc} 1 & \longrightarrow & 2 & = & 2 & & 1 & \longrightarrow & 2 \\ 2 & \longrightarrow & 1 & \longrightarrow & 3 & \iff & 2 & \longrightarrow & 3 \\ 3 & = & 3 & \longrightarrow & 1 & & 3 & \longrightarrow & 1 \end{array}$$

4 Proof for the Theorm

The Futurama question can be written as **can we use product of transpositions to represent** $(12)^{-1}$ **without any transposition appear twice**, two facts is enough for us to show it is always possible.

1) $The\ mapping\ is\ always\ bijective,\ it\ is\ impossible\ for\ a\ body\ contain\ two\ minds\ or\ on\ mind.$

2) $We\ are\ allowed\ to\ add\ more\ new\ people\ in\ our\ process.$

We will give the full idea of proof and leave the proof for you.
The fact 1) implies that now matter how chaotic the switching we have done, the result is a permutation in S_n if we have n people. Every permutation is representable as a product of disjoint cycles (Which you proved by contradiction just now). With the property and some lemmas, we can prove our theorem.

Now we will show the Futurama Theorem in more general cases.

Theorem 4.1 *Let* $\pi = C_1 ... C_r$ *be product of* r *disjoint* $k_i-cycles C_i$ *in* S_n*, with* $k_i \geq 2$ *and* $n = k_1 + ... + k_r$*. We add two more elements* x *and* y*, then* π *can be undone by a product* λ *of* $n+r+2$ *distinct transpositions in* S_{n+2}*, each containing at least one entry in* $\{x, y\}$*.*

Proof (idea) We let $k = k_1$, so C_1 is a k-cycle denoted as $(a_1 ... a_k)$. For the cycle C_1, we define

$$G_1(x) = (a_1 x)(a_2 x)...(a_k x)\ and\ F_1(x) = (a_1 x)$$

Corresponding to each cycle C_i for $i = 1, ..., r$, we define $G_i(x) and F_i(x)$ respectly.
Now we have

$$\sigma = (xy) G_r(x) ... G_2(x)(a_k x) G_1(y)(a_1 x) F_2(y) ... F_r(y)$$

where $\pi\sigma = Id$. □

Proposition 4.1 $I = (12)(2x)(1y)(1x)(2y)(xy)$ *in* S_4

Proposition 4.2 *The product of transpositions in our proof is best possible cannot be replaced by a smaller number*

5 The Shawshank's Redemption

Here's a much more simpler question. Shawshank Prison holds 100 prisoners who were once dangerous but have now changed for the better. The warden decides to give them a chance to start fresh. To organize things, he gives each prisoner a number from 1 to 100, with our main character, Andy, being number 1. Then, he asks them to line up in order, from 1 to 100, and enter a room one by one, starting with Andy. Inside the room, there are 100 boxes numbered from 1 to 100. Before the prisoners enter, the warden randomly places 100 cards, each numbered from 1 to 100, inside these boxes—one card per box.

Each prisoner who enters the room has 50 chances to open boxes. If they find the box containing the card that matches their own number, they will be set free. If not, they will be killed. After opening up to 50 boxes, they must close all the boxes they opened and leave the room. Then, the next prisoner enters and follows the same process. Due to the friendly relationship between the warden and Andy, the warden allows Andy to open all the boxes and then close them. In addition, Andy can choose to exchange the numbers in two of the boxes, either once or not at all. Once the process begins, all prisoners are unable to communicate and enter the room one by one to open and close the box before the next one enters. May I ask if you can come up with an unboxing strategy that allows everyone to be free?

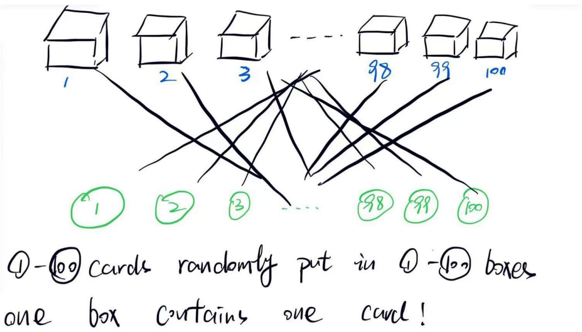

Figure 1: 1-100 cards put in 1-100 boxes

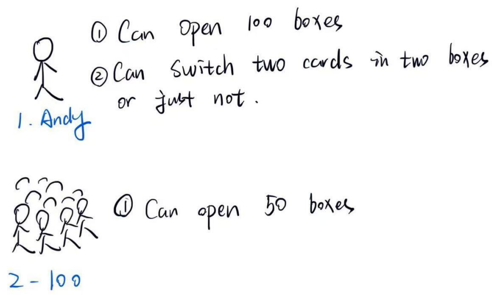

Figure 2: Andy can open all and switch two cards positions, others only open 50

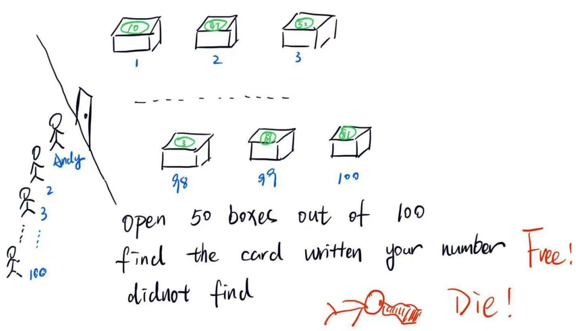

Figure 3: Find the card with your number or dieeeee!!!

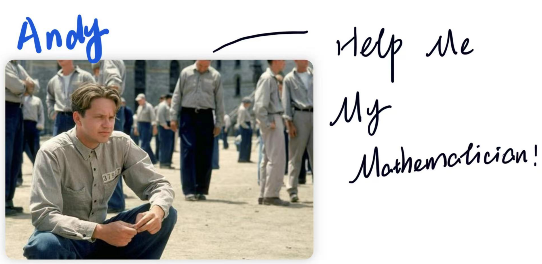

Figure 4: Can you free all the prisoner in Shawshank Prison?

For more details about Futurama theorem, references, extended content, please contact Hongwei.wang22@stuent.xjtlu.edu.cn. Thanks for reading :)