Upperbound for Hausdorff Dimension of Fractals under Symmetry Group

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Contents

Abstract

This paper establishes a rigorous connection between the symmetry groups of self-similar fractals and their Hausdorff dimensions through three fundamental contributions:

We introduce the Hausdorff dimension for the fractals and use iteration functions system to construct the typical Sierpinski triangle. then we discuss the relationship between the symmetric group and fractals, we prove a possible upper bound for the Hausdorff dimension of G-symmetric fractals:

$$\dim_H(F) \leq \frac{\log |G|}{\log(1/s)}$$

where $|G|$ is the order of the symmetry group and s the contraction ratio, with equality holding precisely when the iterated function system (IFS) satisfies the Open Set Condition and exhibits free group action.

1 Introduction: The Coastline Paradox

1.1 Length of Coastline

The concept of distance is quite familiar to us. In daily life, we use distance to describe how far apart two points are. Similarly, by continuously taking points along a boundary and summing their distances, we can calculate the perimeter of an object.

Definition 1. *(Path) Let $P, Q \in \mathbb{R}^n$. A path from P to Q is a continuous map $\gamma : [a, b] \to \mathbb{R}^n$ such that $\gamma(a) = P$ and $\gamma(b) = Q$*

Then by continuously picking points on the path, and summing the length of segments between every two points, we can get total length of the path.

Definition 2. *(length of path) For a path $\gamma : [a, b] \to \mathbb{R}^n$, the length is*

$$length(\gamma) := \sup_{S \subset [a,b]} l_s(\gamma)$$

where

$$l_s(\gamma) := \sum_{k}^{i=0} d(\gamma(t_i), \gamma(t_{i+1}))$$

for some finite sequence (we can consider as times) $a = t_0 < t_1 < t_2 < ... < t_k < t_{k+1} = b$, writing $S = \{t_1, t_2, ..., t_k\} \subset [a, b]$. So our length of path is defined to be the supremum value of all possible finite $S \subset [a, b]$.

Theorem 1. *If $\gamma : [a.b] \to \mathbb{E}^n$ is **continuously differentiable**, i.e γ' exists and continuous, then*

$$length(\gamma) = \int_\gamma ds = \int_a^b |\gamma'(t)| dt < \infty$$

Proof. Needs some analysis □

The theorem of the length of differentiable path is telling us that, for an object whose boundary is continuously differentiable, the perimeter is always bounded. Actually in reality, the perimeter of any physical object is always finite – regardless of whether its boundary curve is differentiable or not. This aligns with our common sense.

Figure 1: The measured length increases with higher measurement precision

However, the **coastline paradox** states that the measured length of a coastline diverges to infinity as the measurement scale $\epsilon \to 0$, revealing its fractal nature. Formally, for a coastline with fractal dimension $D > 1$, the observed length $L(\epsilon)$ scales as:

$$L(\epsilon) \propto \epsilon^{1-D}$$

For some graph that not differentiable, it is hard to measure the length, this phenomenon was first quantified by Mandelbrot in his seminal paper [**?**].

2 Hausdorff Dimension

2.1 Peano curve

In mathematical sense, we all live in 3 dimensional space, every point in our space has three freedoms, or an information (x, y, z) can uniquely determine the position of our point. In such wide definitions, a line is one-dimensional, and a plane is two-dimensional. But in 1890, mathematician Giuseppe Peano found an iteration process which can generate a complete plane by one-dimensional curve.

Peano constructed a space filling curve using a simple initial curve and his iteration system; the limit of this iteration will be the complete two-dimensional plane. This is because if we still use one-dimensional information to determine the point on this graph, all distances from our points to the initial point will be infinte, so it is indeed a two-dimensional plane.

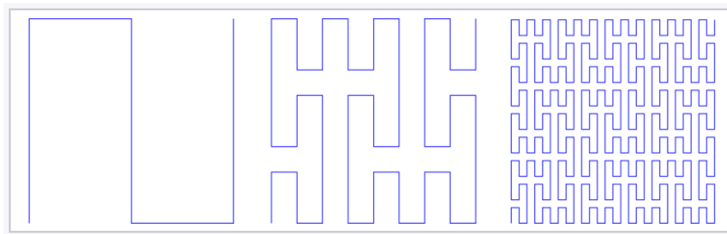


Figure 2: First three iterations of Peano curve

Mathematicians call such pictures **Fractals**– self-similar, continuous but nowhere differentiable.

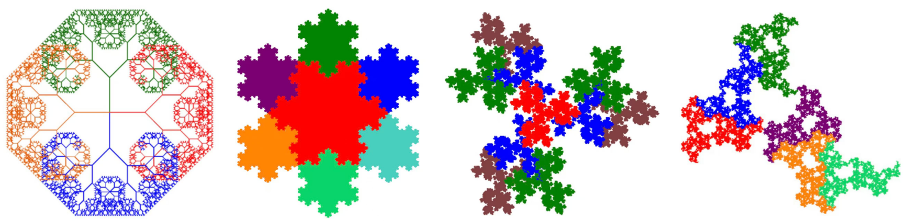


Figure 3: Binary Trees, Koch Snowflake, Koch Curve, McWorter Pentigree

Original definition of dimension gets trouble when used in fractals, all these geometric graphs with finite area but infinite perimeter have non-integer dimension, thus we need a new concept of dimension.

2.2 Hausdorff measure and Hausdorff dimension

Definition 3. *(Hausdorff measure) Let X be a metric space, if $S \subset X$ and $d \in [0, \infty)$, then we have the Hausdorff measure*

$$H^d(S) = \lim_{\delta \to 0} H^d_\delta(S)$$

where

$$H^d_\delta(S) := inf\{\sum_{i=1}^{\infty}(diamU_i)^d : \cup_{i=1}^{\infty} U_i \supseteq S, diamU_i < \delta\}$$

This definition is not that hard to understand. Imagine we are trying to measure the "size" of a weird shape, and the Hausdorff measure is a mathematical tool that asking: "How much d-dimensional stuff fits inside this object?", if d = 1, we are measuring so-called "length", d=2 is "area", and d=3 is "volume"...

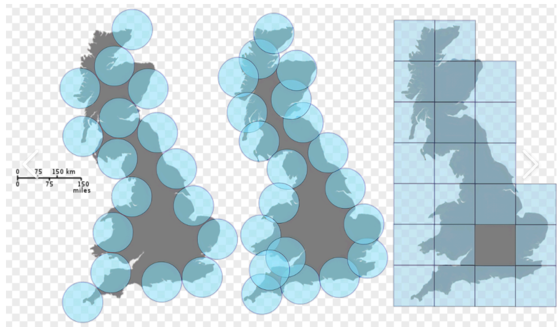


Figure 4: Cover weird shape with diameter of each cover bouded

We have a list of ways to cover our weird shape "S", where each smallest cover has a diameter smaller than a given δ, and we sum all of these covers in a d-dimensional space (the d that appears in the exponent position). The Hausdorff measure is given by the infimum of all these possible values with different covers.

With our Hausdorff measure, we can give the definition of dimension.

Definition 4. *(Hausdorff dimension) The Hausdorff dimension of S is the unique value d where $H^d(S)$ transitions from $+\infty$ to 0, that is*

$$dim_H(S) = inf\{d \geq 0 |\ H^d(S) = 0\} = sup\{d \geq 0 |\ H^d(S) = +\infty\}$$

- *If you measure with **a too small dimension** (e.g., treat a crinkly curve as a 1D line), the "cost" blows up to infinity.*
- *If you measure with **too big a dimension** (e.g., treat the same curve as a 2D surface), the "cost" drops to zero.*
- *The Hausdorff dimension is the **appropriate "d"** where the measurement gives a **finite, meaningful value**—revealing the shape's true complexity (even if it is not an integer).*

2.3 Hausdorff dimension for fractals

But how can we apply our definition to fractals? For a self-similar fractal F, suppose its next iteration F' is constructed by F with number of N and the proportion is r (each small copies of F' is constructed by F times scalar r), the Hausdorff dimension is fixed during the iteration, so that is

$$H^d(F') = NH^d(rF) = Nr^dH^d(F)$$

where rF denotes the small copies of next iteration of F, and F' is constructed by N such one, then we have

$$d = \frac{logN}{log(1/r)}$$

3 Iterated Function Systems

There are numerous ways to generate fractals. Commonly-used methods include iterated function systems, escape-time method, strange attractors, L-systems, etc. We will mainly focus on iterated function systems, the system.

Definition 5. *(Cauchy Sequence and Complete) Let* (X_n) *be a sequence, then we say* (X_n) *is a Cauchy sequence if* $\forall \epsilon > 0, \exists N \in \mathbb{N}$ *s.t.* $\forall n, m \geq N$*, we have*

$$d(X_n, X_m) < \epsilon$$

We say the metric space is complete if every Cauchy sequence in the metric space is converges in it.

Definition 6. *(Contracting Map) Let (X, d) be a metric space. A map* $f : X \to X$ *is a contracting map if* $\exists r \in (0, 1), s.t.$

$$d(f(x), f(y)) \leq rd(x, y),\ \forall x, y \in X$$

Theorem 2. *(Banach fixed-point theorem) Let* X *be a complete metric space and let* $f : X \to X$ *a contracting map, then* $\exists!$ *fixed point for* f *in* X

Proof. See Mth224 □

Theorem 3. *Let M be a complete metric space and* $\{f_1, f_2, ..., f_n\}$ *a family of contracting maps on M. Denote the collection of all nonempty compact subsets of M by* $K(M)$*. Define*

$$\begin{aligned} F : K(M) &\longrightarrow K(M) \\ x &\longmapsto \bigcup_{i=1}^{n} f_n(x) \end{aligned}$$

Then there exists a unique compact subset $X \subseteq M$ $s.t. F(X) = X$ *(Riddle)*

Proof. ... □

Definition 7. *(Iterated Functions System) The set* X *given in* ***Theorem3*** *is called a homogeneous self-similar fractal set, and the family of functions* $\{f_1, f_2, ..., f_n\}$ *is called an iterated functions system.*

4 Fractal

4.1 Sierpinski Gasket constructed by translations

Sierpinski gasket, also called Sierpinski triangle, is a typical fractal constructed by an equilateral triangle subdivided into smaller equilateral triangles recursively.

One of the easiest ways to draw a Sierpinski triangle is to repeat the removal of triangular subsets. First, start with a filled equilateral triangle **S(0)**. Then,

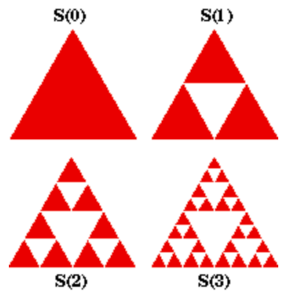
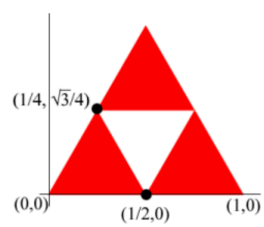


Figure 5: construct Sierpinski triangle by removing triangles

divide into four smaller equilateral triangles treating the midpoints of the three sides of the initial triangle as the new vertices. Then remove the middle one, we have our **S(1)**, and repeat the process to every equilateral triangle to get **S(2)**

We know iterated function systems (IFSs) are good methods of constructing fractals. Fractals are often self-similar. By a proper IFS, the fractals can be constructed very quickly, and the IFS also shows how the fractals are structured.

Looking at image 1, we can see that S(n+1) is actually composed of four S(n) reduced to 1/4 of their size. If we take the lower-leftmost vertex of S(n) as the origin of the coordinate system, then we can make S(n+1) by making a 4-fold reduction in the whole graph, and then duplicate three copies and place them in the corresponding positions



The process we introduce yields the following iterated function system:

$$\begin{aligned}
&\forall x \in S(n): \\
&f_1(x) = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} x \\
&f_2(x) = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} x + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} \\
&f_3(x) = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} x + \begin{bmatrix} 1/4 \\ \sqrt{3}/4 \end{bmatrix}
\end{aligned}$$

The IFS for Sierpinski triangles is an **one to three** mapping, and for each S(n), the size get smaller as

$$det \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} = 1/4$$

4.2 Sierpinski Gasket constructed by rotations

However, for one specific fractal, the iterated function system is not unique, it depends on the way the fractal is constructed. The Sierpinski triangles is a recursive process by removing middle triangles for all equilateral triangles, we can also construct them by rotation because of the symmetry of an equilateral triangle.

We first rotate the triangles with three different angles, 0, $\frac{2\pi}{3}$ and $-\frac{2\pi}{3}$, then keep in mind that each edge of an equilateral triangle is scaled by a factor $\frac{1}{2}$, so we have the IFS:

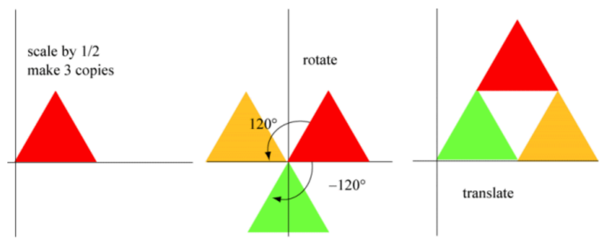


Figure 6: construct Sierpinski triangle by rotations

$$\forall x \in S(n):$$

$$f_1(x) = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1/4 \\ \sqrt{3}/4 \end{bmatrix}$$

$$f_2(x) = \frac{1}{2}\begin{bmatrix} cos(2\pi/3) & -sin(2\pi/3) \\ sin(2\pi/3) & cos(2\pi/3) \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & -1/4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$f_3(x) = \frac{1}{2}\begin{bmatrix} cos(-2\pi/3) & -sin(-2\pi/3) \\ sin(-2\pi/3) & cos(-2\pi/3) \end{bmatrix} x + \begin{bmatrix} 1/4 \\ \sqrt{3}/4 \end{bmatrix}$$

$$= \begin{bmatrix} -1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & -1/4 \end{bmatrix} x + \begin{bmatrix} 1/4 \\ \sqrt{3}/4 \end{bmatrix}$$

This recursive system is also an **one to three** mapping, and

$$det\begin{bmatrix} -1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & -1/4 \end{bmatrix} = det\begin{bmatrix} -1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & -1/4 \end{bmatrix} = 1/4$$

As you can see, no matter what IFS we pick to generate a Sierpinski gasket, the number of mappings and determinants of iteration matrix is fixed. The

Sierpinski gasket is self-similar with 3 non-overlapping copies of itself, each scaled by the factor 1/2. Therefore the Hausdorff dimension of the Sierpinski triangle is

$$d = \frac{log(number\ of\ copies)}{log(1/scalar)} = \frac{log(3)}{log(1/\frac{1}{2})} = \frac{3}{2} \approx 1.58496$$

If we focus on IFS, then the formula to compute Hausdorff dimension of fractal has a iterated form. For iterated functions system $\{f_1, f_2, ..., f_n\}$, $f_i(x) = A_i x + b_i$, we define $\alpha := detA_1 = detA_2 = ... = detA_n$(their determine should be the same), the Hausdorff dimension d of the fractal construct by this IFS is

$$d = \frac{log(1/n)}{log(\sqrt{\alpha})}$$

This formula is ideally the same as our original one.

One thing should be highlighted here is that Sierpinski triangle does not derive from its primary constituent being an equilateral triangle, we can use any element we want as the S(0), as long as we have a proper IFS based on the properties of Sierpinski triangle, or any other fractal we want, we can generate it, no matter how weird the original element is.

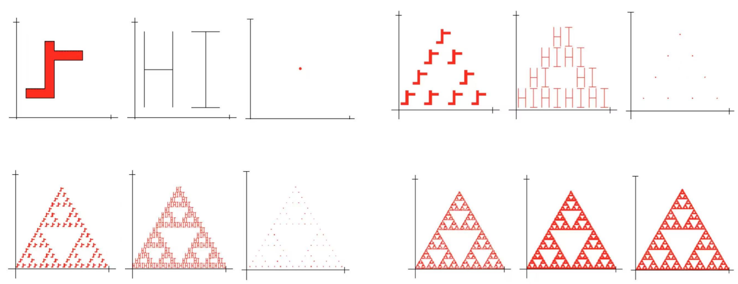


Figure 7: Three different trivial elements under same iterated functions system

The figure above shows the iteration results of different shapes under the same IFS. It demonstrates that despite having irregular initial shapes, even a dot, the final results converge to the same pattern when processed through identical IFS transformations

5 Symmetry Group Order and the Hausdorff Dimension Upper Bound

5.1 Symmetry groups and Stabilisers of fractals

It is essential for us to discover a way to describe the hausdorff dimension by the shape of ths fractal itself. All fractals are self-similar, by definition. That means the shape of the fractal hide in itself, in such way we can use symmetry group to discover the dimension of fractal, specifically, dihedral group.

Definition 8. *(Dihedral Group) We define the n^{th} dihedral group for a set S as*

$$D_n := \{T \in GL_2(\mathbb{R}) | T(P_n) = P_n\}$$

where P_n is regular n-gon with center at origin, and we say D_n is the symmetry group of the fractal S is the regular n-gon is replaced by the fractal (i.e. $T(S) = S$).

Take Sierpinski triangle as example, D_3 is the symmetry group, where $D_3 = \{e, \varphi_1, \varphi_2, \varphi_3, \sigma_1, \sigma_2\}$, contains two rotations and three reflection, every elements in D_3 acts on Sierpinski triangle still remain the shape of it.

Definition 9. *(Stabilisers) G is a group acts on set X. The stabiliser of a point $x \in X$ is*

$$Stab_G(x) = \{g \in G | gx = x\} \leq G$$

Under such definition, when focusing on small copy of Sierpinski triangle, we only got two elements of D_3 as the stabiliser subgroup.

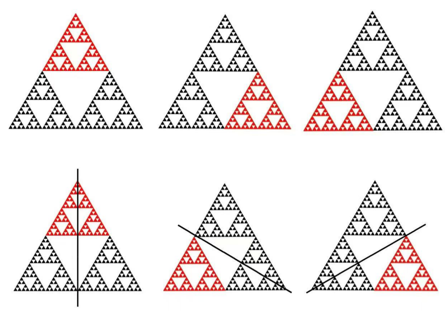


Figure 8: First row: Sierpinski triangles applied $e, \sigma_1, \sigma_2 \in D_3$, (i.e. Identity, rotation $\frac{2\pi}{3}$ *and* $\frac{4\pi}{3}$). Second row: Sierpinski triangles applied $\varphi_1, \varphi_2, \varphi_3 \in D_3$, (i.e. three reflections)

For Sierpinski triangle, when we focus on one specific small copy, we have $|Stab_G| = 2$. And we know fractals are all self-similar, which means other stabiliser of small copies have the same order, in such way we can calculate the number of each iterations with

$$N = \frac{|G|}{|Stab_G|} = \frac{6}{2} = 3$$

5.2 Upperbound for Hausdorff dimension

The relationship between symmetry groups and fractal dimensions becomes clearer after examining a broader range of examples. Table1 extends classical cases to other fractals representing diverse symmetry groups.

Table 1: Extended comparison of symmetry-dictated fractal parameters

Fractal	G	$\lvert G\rvert$	$\lvert H\rvert$	N	s	$\dim_H(F)$
Koch snowflake	D_6	12	2	4	1/3	$\log 4/\log 3 \approx 1.262$
Sierpiński gasket	D_3	6	2	3	1/2	$\log 3/\log 2 \approx 1.585$
Cantor set	D_1	2	1	2	1/3	$\log 2/\log 3 \approx 0.631$
Pentagonal flake	D_5	10	2	5	1/3	$\log 5/\log 3 \approx 1.465$

A fundamental patterns emerge from this extended dataset: The order of symmetry group $|G|$ correlates with fractal complexity. Higher-order groups (e.g., O_h with $|G| = 48$ for the Menger sponge) permit more intricate replication structures ($N = 20$), leading to higher Hausdorff dimensions (≈ 2.727). The dimensional scaling follows:

$$\dim_H(F) \propto \frac{\log(|G|/|H|)}{\log(1/s)}$$

where the proportionality constant depends on the geometric embedding. For some regular fractals, which is symmetric and well-constructed, the proportion sign can be equality under some conditions.

Definition 10. *(G-symmetric fractals) We say the fractals is G-symmetric if the fractals is* ***self-symmetric*** *and invariant under elements of G*

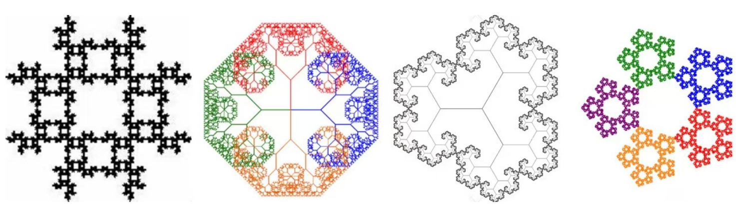


Figure 9: Some self-symmetric fractals

Theorem 4 (Group-Order Dimension Bound)**.** *For a self-similar fractal F generated by a G-symmetric iterated function system with contraction ratio $s \in (0, 1)$, its Hausdorff dimension satisfies:*

$$\dim_H(F) \leq \frac{\log |G|}{\log(1/s)},$$

Proof. By the orbit-stabilizer theorem, the number of similarity copies N satisfies:

$$N = \frac{|G|}{|H|} \leq |G|$$

since $|H| \geq 1$ for any stabilizer subgroup $H \leq G$, and the Hausdorff dimension is given by:

$$\dim_H(F) = \frac{\log N}{\log(1/s)} \leq \frac{\log |G|}{\log(1/s)}$$

Equality Condition: Equality requires $N = |G|$, which occurs precisely when $|H| = 1$ for all stabilizers, i.e., G acts freely. □

6 Summary

Our paper introduce the hausdorff dimension and give the formula to calculate the dimension of fractals

$$d = \frac{logN}{log(1/r)}$$

Where N is the number of copies for next iteration and the $r \in (0, 1)$ is the scalar we time each iteration.

Then we introduce the iteration functions systems and take Sierpinski triangle as an example, given two different IFS to constructed the same fractals. In IFS eyesight, the fractals dimension can be written as

$$d = \frac{log(1/n)}{log(\sqrt{\alpha})}$$

Where n is the number of our iteration functions, and α is the determine value of matrics in IFS. At last we try to use algebraic way to calculate the hausdorff dimension, by the self-similar properties of fractals, we find the symmetric group of fractals may provied an upperbound of the dimension, with the orbit-stabiliser theory, we imply an inequality

$$\dim_H(F) = \frac{\log N}{\log(1/s)} \leq \frac{\log |G|}{\log(1/s)}$$

This theory is only valid for self-symmetric fractals. By identifying the symmetry group of a self-symmetric fractal, we can derive the upper bound of its Hausdorff dimension. For stabilizer subgroups with partial integer orders, the Hausdorff dimension can be precisely calculated. Under different iterated function systems (IFS), the dimension of the orbit subgroup may be non-integer, which is a common occurrence. This leaves room for future expansion and discussion.

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