Categorical Algebra and Homotopy Theory

Hongwei.Wang[1]

[1] Xi'an Jiao Tong Liverpool Univercity

2025.10.14

Contents

1 Introduction

...

2 Category Theory

2.1 Category

Definition 2.1 *(Category) A* ***category*** $\mathscr{C}$ *consists of a collection* $ob(\mathscr{C})$ *of* ***objects****, and* $\forall A, B \in ob(\mathscr{C})$*, there is a collection* $\mathscr{C}(A, B)$ *of* ***morphisms*** *from A to B. With following three axioms satisfied:*
1. ***Compostion****:* $\forall A, B, C \in ob(\mathscr{C})$*, if* $f \in \mathscr{C}(A, B)$ *and* $g \in \mathscr{C}(B, C)$*, then there is a function* $g \circ f \in \mathscr{C}(A, C)$ *called compostion of f and g.*
2. ***Associativity****:* $(h \circ g) \circ f = h \circ (g \circ f),\ \forall f \in \mathscr{C}(A, B),\ g \in \mathscr{C}(B, C),\ h \in \mathscr{C}(C, D)$*.*
3. ***Identity laws****:* $\forall A \in ob(\mathscr{C}),\ \exists 1_A \in \mathscr{C}(A, A)$*, called the identity of A. And* $\forall f \in \mathscr{C}(A, B)$*, we have* $f \circ 1_A = f = 1_B \circ f$*.*

Example 2.1 *(Category of Sets:* ***Set****) For a category* $\mathscr{C}$*, where* $ob(\mathscr{C})$ *is a collection of sets, here we consider each set as one object, no matter the cardinality of it. Given two set A and* $B \in ob(\mathscr{C})$*, the mapping or morphism between two sets is exactly a function from A to B. Together with all the sets we have and the functions between each two sets we call them* ***Set****, the category of set.*

Example 2.2 *(Category of Groups and Rings:* ***Grp*** *and* ***Ring****) We have a collection of groups, and a morphism between every two given group G and H which is so-called group homomorphism. Then all these groups together with the group homomorphisms are called category* ***Grp*** *of groups. Similarly, there is a category* ***Ring*** *of rings and ring homomorphisms.*

Example 2.3 *(Category of Vector Spaces over field k:* $\mathbf{Vect}_k$*) For a field k,* $\mathbf{Vect}_k$ *consists the vector fields over k and the mapping between two vector spaces H to W which will be the k linear transformations from H to W, i.e.* $\mathcal{L}(H, W)$*.*

Example 2.4 *(Category of Topological Spaces:* ***Top****) There is a collection of topological spaces and the mapping between topological spaces are continous maps. Together the topological spaces and maps are called* ***Top****.*

The 'Category' is not a unique object but a structure of mathematical objects. For instance, Group is something in a set satisfy associativity and there is an identity and inverce under specific operation, the set in definition of group can be anything, they could be functions, person, computer program, formula for Rubik's cube... So similarly, here the category of some mathematical object is not represent all these object in the collection. You can construct any category you want with few objects as long as you give the morphisms and they satisfy the axioms.

Take **Set** as example, the category of set not necessarily contain all sets, if you construct a category with limited number of sets, it still called **Set**.

Example 2.5 *(Category of nothing:* $\emptyset$*) There is a collection of nothing and no morphisms between nothing, these called empty category* $\emptyset$*.*

Example 2.6 *(Category of one object:* ***1****) There is a category* ***1*** *with only one object in the collection and only Identity map.*

Example 2.7 *(Discrete Category) A category* $\mathscr{C}$ *is discrete if* $\forall A, B \in ob(\mathscr{C}), \mathscr{C}(A, B) = \emptyset$*. This does not mean there is no mapping in* $\mathscr{C}$*, notice that* $1_A \in \mathscr{C}(A, A),\ \forall A$*.*

Example 2.8 *(One object category constructed by a group) A group actually is a one object category. Differ this from category* ***1*** *that* ***1*** *has only identity but for group one element in it is a morphism. Let us put in a clear way. We have a group $G = \{e, g_1, g_2, ...\}$, consider a category $\mathscr{C}$ that $ob(\mathscr{C}) = G$, the identity morphism in $\mathscr{C}(G, G)$ is actually $e \in G$:*

$$e(G) = e \cdot G = \{e \cdot e, e \cdot g_1, e \cdot g_2 ...\} = \{e, g_1, g_2, ...\} = G$$

and $\forall g \in G$, $g(G) \subseteq G$ from the closure of group structure, if $|G| < \infty$, then $g(G) = G$, $\forall g \in G$. The corresponding table below helps you to understand the isomorphism between mathematical structure.

Category $\mathscr{C}$ with single object A	**Group G**
Maps in $\mathscr{C}$	Elements in G
$\circ$ in $\mathscr{C}$	$\cdot$ in G
$1_A \in \mathscr{C}(A, A)$	$e_G \in G$

We remark that the category of one mathematical object is a collection of some structural objects not necessarily all the objects, and we provide a example say one object category of group.

Now we put our focus into the morphisms in category, given A and B as object of category $\mathscr{C}$, the mapping in $\mathscr{C}(A, B)$ should not necessarily be so-called functions or transformations, we name the morphisms as transformations it is because for $f \in \mathscr{C}(A, B)$, we have $f(A) = B$, gives us feeling that the morphism f trans A into B.

We should consider it more abstract, the type of $f(A) = B$ is actually a one directional relation of A and B. If f is not some machine but a not comprehensive statement, for example: $f :=$ 'is bigger than', then $f(A) = B$ is a full statement:

$$f(A) = B \Longleftrightarrow A \longrightarrow B \Longleftrightarrow A \textit{ is bigger than } B$$

Consider mapping as relation between different objects is one of core idea in category theory, it is a great abstraction and acrroding to this we can find many isomorphism between different mathematical structures.

Definition 2.2 *(opposite category) a category noted $\mathscr{C}^{op}$ is said to be the opposite or dual category of given category $\mathscr{C}$, it has exactly the same object with all the arrows in $\mathscr{C}$ reversed, thas is:*

$$ob(\mathscr{C}^{op}) = ob(\mathscr{C}) \;\; \textit{and} \;\; \mathscr{C}^{op}(B, A) = \mathscr{C}(A, B)$$

2.2 Functor

Definition 2.3 *(Functor) We say a map of categories $\mathcal{F} : \mathscr{C} \longrightarrow \mathscr{D}$ is a functor, if it sends every A in $ob(\mathscr{C})$ to a $\mathcal{F}(A)$ in $ob(\mathscr{D})$ and a morphism $f : A \longrightarrow B$ of $\mathscr{C}$ to a morphism $\mathcal{F}(f) : \mathcal{F}(A) \longrightarrow \mathcal{F}(B)$ of $\mathscr{D}$, while satisfies two axioms that*

$$\mathcal{F}(id_A) = id_{\mathcal{F}(A)} \;\; \textit{and} \;\; \mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$$

where $A \in ob(\mathscr{C})$ and $f, g \in mor(\mathscr{C})$

The fundamental group map $\pi_1(*, *)$ can be considered as a functor from the category of topological space with the basepoint **Top*** to **Grp**. For any topological space X with a basepoint x specified, the functor send the pair $(X, x) \in ob($**Top***$)$ to its fundamental group $\pi_1(X, x) \in$ **Grp**, with give us the algebric structure of the loops start at basepoint x in space X. In this case any $f \in$ **Top***(X, Y) not only a continous map from X to Y but also a basepoint-preserving $f : (X, x) \longrightarrow (Y, y)$, with its image under the functor $\pi_1(f) : \pi_1(X, x) \longrightarrow \pi_1(Y, y)$.

For examples apart from algebric topology, we have forgetful functors from **Grp** to **Set** which just by its name, the group forgets its structure under the operation but keep its members as a set. And free functors can be considered as dual functor of forgetful, send a set to a group with an operation and add more elements in the set to make it a group.

One type a functor is widely used in categorical language, for a locally small category $\mathscr{C}$ and $A \in ob(\mathscr{C})$, we have $H^A = \mathrm{Hom}(A, -) : \mathscr{C} \longrightarrow \mathbf{Set}$. The morphism functor send every element X in the category to the morphism set $\mathrm{Hom}(A, X)$, which is the set of all morphisms from A to X. And the morphism map under functor is $H^A(g) = \mathrm{Hom}(A, g) : \mathrm{Hom}(A, X) \longrightarrow \mathrm{Hom}(A, Y)$, simply by $f \longmapsto g \circ f$ for all $f : X \longrightarrow Y$.

But if the postion of given $A \in ob(\mathscr{C})$ switch then everything changed. For similar morphism functor $H_A = \mathrm{Hom}(-, A)$, the narrow preserving diagram make no sense, the image $H_A(g)$ cannot be defined for certain $g \in \mathrm{Hom}(X, A)$, so the functor is actually defined on the opposite category of $\mathscr{C}$, that is, $H_A = \mathrm{Hom}(-, A) : \mathscr{C} \longrightarrow \mathbf{Set}$ (See diagram below). This gives us the motivation to define the special functor on the opposite category.

$$
\begin{tikzcd}
X \arrow[r, "H^A"] \arrow[d, "f"'] & \mathrm{Hom}(X, A) \arrow[d] \\
Y \arrow[r, "H^A"] & \mathrm{Hom}(Y, A)
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "f"'] & A \\
Y \arrow[ur, "cannot\ define"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
X \arrow[r, "H^A"] & \mathrm{Hom}(X, A) \arrow[d] \\
Y \arrow[u, "f'"] \arrow[r, "H^A"] & \mathrm{Hom}(Y, A)
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "g"] & A \\
Y \arrow[u, "f'"] \arrow[ur, "g \circ f"'] &
\end{tikzcd}
$$

Definition 2.4 *(contravariant functor) Let $\mathscr{C}$ and $\mathscr{D}$ be categories, a functor $\mathscr{C}^{op} \longrightarrow \mathscr{D}$ is said to be the contravariant functor from $\mathscr{C}$ to $\mathscr{D}$.*

2.3 Natural transformations

We have defined the categories and functors as the mapping of two categories so far, the definition of functors is actually equivalent to say the diagram below commutes.

$$
\begin{tikzcd}
\mathscr{C} \arrow[d, shift right=1.5ex, "\mathcal{F}"'] \arrow[d, "\mathcal{G}"'] \arrow[d, shift left=1.5ex, "\mathcal{H}"'] \\
\mathscr{D}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & X \arrow[dl, "\mathcal{F}"'] \arrow[dr, "\mathcal{G}"] \arrow[drr, "\mathcal{H}"] \arrow[dd, "f"] & & \\
\mathcal{F}(X) \arrow[rr, dotted, dash] \arrow[dd, "\mathcal{F}(f)"] & & \mathcal{G}(X) \arrow[r, dotted] \arrow[dd, "\mathcal{G}(f)"] & \mathcal{H}(X) \arrow[dd] \\
 & Y \arrow[dl, "\mathcal{F}"] \arrow[dr, "\mathcal{G}"'] \arrow[drr, "\mathcal{H}"] & & \\
\mathcal{F}(Y) \arrow[rr, dotted, dash] & & \mathcal{G}(Y) \arrow[r, dotted] & \mathcal{H}(Y)
\end{tikzcd}
$$

So it is so natural to define a new mapping to fill in the gaps in the dashed line at the base of the triangle. To be precise, we have to make the bottom rectangular commutes. Such mapping between two functors comes up in a natural way thus we call it natural transformations.

Definition 2.5 *Given two categories $\mathscr{C}$ and $\mathscr{D}$, and two functors $\mathcal{F}, \mathcal{G} : \mathscr{C} \longrightarrow \mathscr{D}$, the natural transformation is a map of functors $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$, which consists a morphism $\alpha_X : \mathcal{F}(X) \longrightarrow \mathcal{G}(X)$ for all $X \in ob(\mathscr{C})$ such that for all $f : X \longrightarrow Y$, the following diagram commutes*

$$\begin{tikzcd}
\mathcal{F}(X) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "\alpha_X"] & \mathcal{F}(Y) \arrow[d, "\alpha_Y"] \\
\mathcal{G}(X) \arrow[r, "\mathcal{G}(f)"] & \mathcal{G}(Y)
\end{tikzcd}$$

2.4 Limits and colimits

Now we introduce two of the most important and powerful concepts in category theory. Limits and colimits give us a way to discover the mathematical structure universally and uniquely. Both of them are defined after a special functor.

Definition 2.6 *(Category-shaped diagram) Let $\mathscr{C}$ and $\mathscr{D}$ be category and small category. The functor $\mathcal{F} : \mathscr{D} \longrightarrow \mathscr{C}$ is a $\mathscr{D}$-shaped diagram in $\mathscr{C}$.*

We have the category $\mathscr{D}[\mathscr{C}]$ called $\mathscr{D}$-shaped diagram category in $\mathscr{C}$ where $\mathrm{Hom}(\mathcal{F}, \mathcal{F}')$ are the natural transformations. And our limits will be defined in the image of one specific $\mathscr{D}$-shaped diagram in the category $\mathscr{C}$. We already know for any two $D, D' \in \mathscr{D}$, the image of $\mathcal{F}$ make the rectangular commutes. Moreover, for fixed $A \in \mathscr{C}$, it should be commutative with every image of elements in $\mathscr{D}$ under the functor.

Definition 2.7 *(Cone) A cone on the $\mathscr{D}$-shaped diagram functor $\mathcal{F}$ consists a vertex $A \in ob(\mathscr{C})$ and the family of maps in mor($\mathscr{C}$) $f_i : A \longrightarrow \mathcal{F}(D_i)$ where $i \in I$, $D_i \in ob(\mathscr{D})$, such that $\forall g_i : D_i \longrightarrow D_{i+1}$ in $\mathscr{D}$, the cone diagram commutes:*

$$\begin{tikzcd}
 & & & A \arrow[dlll, "f_1"'] \arrow[dll, "f_2"] \arrow[dl, "f_3"] \arrow[d, "f_4"] \arrow[dr, "f_5"] \arrow[drr, "f_{k-1}"] \arrow[drrr, "f_k"] & & & \\
\mathcal{F}(D_1) \arrow[r, "\mathcal{F}(g_1)"'] & \mathcal{F}(D_2) \arrow[r, "\mathcal{F}(g_2)"'] & \mathcal{F}(D_3) \arrow[r, "\mathcal{F}(g_3)"'] & \mathcal{F}(D_4) \arrow[r, "\mathcal{F}(g_4)"'] & (D_5) \arrow[r, dotted] & \mathcal{F}(D_{k-1}) \arrow[r, "\mathcal{F}(g_{k-1})"'] & \mathcal{F}(D_k)
\end{tikzcd}$$

We denote this cone as $(f_i : A \longrightarrow \mathcal{F}(D_i))_{i \in I}$. Such cone-shaped structure is easy to find in any category and the number of such cones is large. So what we want to find is a so-called best cone which can represented as the top cone of every cone and give us an universal structure of the category.

Definition 2.8 *(Limits and colimits) The limits of $\mathcal{F}$ is a vertex of the cone (φ_i: $\lim\mathcal{F} \longrightarrow \mathcal{F}(D_i))_{i \in I}$ satisfying that for any cone of $\mathcal{F}$: $(f_i : A \longrightarrow \mathcal{F}(D_i))_{i \in I}$, there is a unique map in mor($\mathscr{C}$) $\tilde{f} : A \longrightarrow \lim\mathcal{F}$ such that $\varphi_i \circ \tilde{f} = f_i$, $\forall i \in I$. The colimits is the dual of limits which is defined by reversing arrows. These are equivalent to say such diagram commutes for the simple case of any two $D_1, D_2 \in \mathrm{ob}(\mathscr{D})$:*

$$\begin{tikzcd}
 & \lim\mathcal{F} \arrow[ddl, "\varphi_1"'] \arrow[ddr, "\varphi_2"] & \\
 & A \arrow[u, "\tilde{f}"'] \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\
\mathcal{F}(D_1) \arrow[rr, "\mathcal{F}(d_1)"'] & & \mathcal{F}(D_2)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathrm{colim}\mathcal{F} \arrow[d, "\tilde{g}"] & \\
 & A & \\
\mathcal{F}(D_1) \arrow[uur, "\phi_1"] \arrow[ur, "g_1"] \arrow[rr, "\mathcal{F}(d_1)"'] & & \mathcal{F}(D_2) \arrow[uul, "\phi_2"'] \arrow[ul, "g_2"']
\end{tikzcd}$$

In categorical way, for one $\mathscr{D}$-shaped diagram functor $\mathcal{F}$, the image category $\mathscr{C}$ can considered as the category with objects are the vertex of cones defined on the functor $\mathcal{F}$, and $\lim\mathcal{F}$ is the terminal object while $\mathrm{colim}\mathcal{F}$ is the initial in such category.

3 The fundamental groupoid

3.1 Homotopy equivalence

For the category of topological space with basepoint $\mathbf{Top}^*$ we mentioned before, its objects consists a pair (X, x). The morphisms in $\mathbf{Top}^*$ only continously send space X to Y but also preserve the chosen basepoint x to y. In such case, the fundamental group is a functor $\pi_1(*, *) : \mathbf{Top}^* \longrightarrow \mathbf{Grp}$, where $\mathbf{Grp}$ is the category of groups with group homomorphisms. For a topological space take the homotopy class as the equivalence relation, a two path from x to y in space X should be considered as one if they in same homotopy class. But a path also can be considered as a morphism from x to y, we are saying in a categorical way, the morphism in category is equivalent to a path in topological space, that is:

$$\mathrm{Hom}_{\mathbf{Top}^*}(x, y) \sim \{\gamma \in C^0([0,1], X) | \gamma(0) = x, \gamma(1) = y\}$$

Under this intuition, for any category $\mathscr{C}$, take all the morphisms as paths, we define the homotopy category $h\mathscr{C}$ as the category with the same objects as $\mathscr{C}$ but with morphisms under the homotopy classes of maps.

Definition 3.1 *(Homotopy category) For any category $\mathscr{C}$, the homotopy category $h\mathscr{C}$ is a category which*

$$\mathrm{ob}(h\mathscr{C}) = \mathrm{ob}(\mathscr{C}) \quad \textit{and} \quad \mathrm{mor}(h\mathscr{C}) = \mathrm{mor}(\mathscr{C})/[\textit{homotopy class}]$$

For the unbased topological space category $\mathscr{U}$, recall a homotopy equivalence of topological space $X, Y \in \mathrm{ob}(\mathscr{U})$ is equivalent to the existence of $f \in \mathrm{Hom}(X, Y)$ and $g \in \mathrm{Hom}(Y, X)$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. That is actually saying a homotopy equivalence in $\mathscr{U}$ is an isomorphism in $h\mathscr{U}$.

Proposition 3.1 *A group isomorphism $f_* \in \mathrm{Hom}_{\boldsymbol{Grp}}(\pi_1(X, x), \pi_1(Y, f(x)))$ with independence of the choice of basepoint x is induced by the homotopy equivalence $f \in \mathrm{Hom}_{\mathscr{U}}(X, Y)$.*

Proof Let □

3.2 Fundamental groupoid

Now we introduce a new mathematical construction, the groupoid. The name groupoid is come from the algebric structure group but more general. A group can consider as a one object category in the very first example in category, for a singleton $\{*\}$ we have all its automorphisms as a group, which means the category with single object $*$ a point and all the elements in group as the morphisms in category. Groupiod is defined similar to this but on more than one objects.

Definition 3.2 *(Groupiod) A groupiod $(G, \circ)$ is a category which every morphism is isomorphism.*

It is not hard to see a group is a groupiod, but the reverse is not. More clearly for $A \in \mathrm{ob}(G)$ where $G \in \mathbf{GP}$ we have $\mathrm{Hom}_G(A, A) = Aut(A)$ is a group and a one object category. One fact is obvious that the category $\mathbf{GP}$ is a category of categories, so every morphism is a functor.

Recall the definition of fundamental group, which is strictly dependent on the choice of base point x_0. Somehow we know for a path-connected space X we have $\pi_1(X, x_0) \cong \pi_1(X, y_0)$, but this is according to the choice of the path from x_0 to y_0, thus we still cannot say 'the fundamental group of X' but 'the fundamental group of X on the point x_0'.

This comes out as the motivation to define a new fundamental structure on the generalization of groups.

Definition 3.3 *(Fundamental groupiod) For a topological space X and $x, y \in X$. We have $Path(x, y) := \{\gamma \in C^0([0,1], X) | \gamma(0) = x, \gamma(1) = y\}$. The Fundamental groupiod of a topological space X is a category $\prod(X)$ where $\mathrm{ob}(X) = X$, and $\mathrm{Hom}(x, y) = \mathrm{Path}(x, y)/\sim$, where $\sim$ is the equivalence relation under homotopy class respect to the chosen two points.*

You should verify it quickly and easily that

$$\mathrm{Hom}_{\prod(X)(x_0, x_0)} = \pi_1(X, x_0)$$

The construction of fundamental groupiod will give us convenient way to study the structure of a topological space. Any continous map $f : X \longrightarrow Y$ induces a group homomorphism $f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, f(x_0))$ while we have to pick $f(x_0)$ properly. However the advantages of groupiods gives us a induced functor $\prod(f) : \prod(X) \longrightarrow \prod(Y)$ and this is more clean and formal. The diagram below show the difference between fundamental groups and fundamental groupiods. Both π_1 and $\prod$ are functors but groupiod induces functor (It is ok that you may think $\pi_1(f)$ is also a functor because group is one object category, but this brings us nothing new to describe the structure of space and we still have the hindrence to find the basepoint).

$$\begin{tikzcd}
\mathbf{Top}^* & \mathbf{Grp} & \{*\} \in \mathrm{ob}_{\mathbf{Top}^*} \\
X \arrow[r, "\pi_1"] \arrow[d, "f"] & \pi_1(X, x_0) \arrow[d, "\pi_1(f)"] & x \arrow[l, "\pi_1"'] \arrow[d, "f"] \\
Y \arrow[r, "\pi_1"] & \pi_1(Y, f_0) & f(y) \arrow[l, "\pi_1"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{Top}^* & \mathbf{GP} \\
X \arrow[r, "\prod"] \arrow[d, "f"] & \prod(X) \arrow[d, "\prod(f)"] \\
Y \arrow[r, "\prod"] & \prod(Y)
\end{tikzcd}$$

4 A generalization for van Kampen Theorem

4.1 Formal van Kampen Theorem

So far we know some fundamental groups of simple topological space. The idea is that we shall consider different topological space are constructed by those simple objects and figure out a formula to compute those fundamental group from what we already know.

Before we get into the theorem, let us first image what the fundamental group of space X will be if it can be decomposed into two open sets both path-connected and contain the basepoint. Take the shape ∞ as example it can be decomposed into the union of two S^1 and the basepoint is the intersection of two S^1s. At first guess $\pi_1(\infty)$ somehow should be the product group $\mathbb{Z} \times \mathbb{Z}$ or the direct sum $\mathbb{Z} \oplus \mathbb{Z}$. But both of these is commutative, contradicts to the fact that once you get into a S^1 from basepoint you must finish the loop before you get into the other S^1. In mathematical words, if a^2ba represents go twice conter-clockwise in left S^1 then go right S^1 and left again, this path should be distinct and never be the same as a^3b.

So in general, given a collection of open sets A_α which our topological space X can be decomposed to, we wish to construct a single group containing all $\pi_1(A_\alpha)$ as subgroups with non-commutative structure. That introduce the free product of groups.

Definition 4.1 *(categorical free product of groups) Let $(G_i)_{i \in I}$ be a family of groups. The free product $*_{i \in I} G_i$ is a group satisfying the universal property:*

$$\exists homomorphisms\ \iota_j : G_j \longrightarrow *_{i \in I} G_i,\ s.t.$$
$$For\ any\ group\ H\ and\ any\ family\ of\ homomorphisms\ f_j : G_j \longrightarrow H,$$
$$\exists! homomorphism\ \phi : *_{i \in I} G_i \longrightarrow H\ making\ the\ diagram\ commute\ \forall j \in I$$

$$\begin{tikzcd}
G_j \arrow[rr, "\iota_j"] \arrow[dr, "f_j"'] & & *_{i\in I}G_i \arrow[dl, "\phi"] \\
& H &
\end{tikzcd}$$

The definition is under categorical language and it may be abstract. For a formal definition we have to construct the free product of any family of groups by defining the 'words' in the product. Here is an example for an equivalent constructive definition

Definition 4.2 *(formal free product of groups) For a family of groups $(G_i)_{i\in I}$, a word is a finite sequence $s_1 s_2 ... s_n$, where $s_k \in G_i \backslash \{e\}$ for some $i \in I$. A word is said be reduced if any two adjacent letters $s_i s_{i+1}$ belong to different groups. The free product $*_{i\in I}G_i$ is the group that elements are reduced words concatenated by the following reduction rules:*
i). If adjacent letters belong different groups, simply concatenate.
ii). If adjacent $s_i s_{i+1}$ from the same group G_i, replace them with the result of their product $s_i \circ s_{i+1} \in G_i$ where $\circ$ is the operation for specific G_i.
iii). Remove the product if $s_i \circ s_{i+1} = e_i \in G_i$.

The two definitions are exactly the same but with different mathematical language. All is trying to say the free product group is also a group and it contains all the elements of the groups in our family. It allows us to represent the fundamental group by decomposed the space into many open pieces and remain the non-commutative structure.

Theorem 4.1 *(van Kampen) If X is decomposed as the union of path-connected open sets A_α, with each containing the basepoint $x_0 \in X$ and $A_{\alpha_i} \cap A_{\alpha_j}$ is path-connected for any two, then the homomorphism $\phi : *_\alpha \pi_1(A_\alpha) \longrightarrow \pi_1(X)$ is surjective.*

4.2 Categorical van Kampen theorem

One of our purpose is to introduce homotopy theory from categorical language, a potential way to do that is to find the equivalence between different mathematical description to same topological object or theorem. The van Kampen theorem is a good example.

Let $\mathcal{O} = \{U_i\}_{i\in I}$ be a cover of the space X where every U_i is path connected open subsets and $\cap_{i\in K\subseteq I}^{|K|<\infty} U_i \in \mathcal{O}$, which means the intersection of finitely many subsets in $\mathcal{O}$ is again in $\mathcal{O}$. $\mathcal{O}$ can be considered as a category with objects are those open subsets and morphisms are the inclusions between sets. In such way the fundamental groupiod functor restricted to the space and maps in $\mathcal{O}$ sends us to the category of groupiod, that is, $\prod|_{\mathcal{O}} : \mathcal{O} \longrightarrow \mathbf{GP}$. And what van Kampen theorem says in most general categorical words is the fundamental groupiod is the initial objects in the category of all vertex of cones on the groupiod functor.

Theorem 4.2 *(General van Kampen) The groupiod $\prod(X)$ is the colimit of $\mathcal{O}$-shaped diagram in **GP**, in mathematical words:*

$$\prod(X) \cong \mathrm{colim}_{U_i\in\mathcal{O}} \prod(U_i)$$

Proof To construct a proper proof, we have to verify the universal property of colimits. Fundamental groupiod functor restricted on the cover is a map between two categories $\mathscr{O} \longrightarrow \mathbf{GP}$, which is a $\mathscr{O}$-shaped diagram. The theorem is saying $\prod(X)$ is $\mathrm{colim}(\prod|_{\mathscr{O}})$. We have to show, $\forall G \in \mathbf{GP}$ and the family of $\phi_i : \prod(U_i) \longrightarrow G$, $\exists! \Phi : \prod(X) \longrightarrow G$ such that $\phi_i = \Phi \circ \iota_i$ for every i, where ι_i is the morphism induced by the inclusions of subsets in category $\mathscr{O}$ under functor $\prod|_{\mathscr{O}}$.

$$\begin{tikzcd}
\prod(U_1) \arrow[rr, "\prod(\hookrightarrow)"] \arrow[dr, "\iota_1"] \arrow[ddr, "\phi_1"'] & & \prod(U_2) \arrow[dl, "\iota_2"'] \arrow[ddl, "\phi_2"] \\
& \operatorname{colim} \prod|_{\mathscr{O}} = \prod(X) \arrow[d, dashed, "\exists!\ \Phi"] & \\
& G &
\end{tikzcd}$$

Our proof includs two steps. First we have to define the morphism $\Phi : \prod(X) \longrightarrow G$ then show it is unique. Be careful that we are working in the category of groupiod, which means any morphism between two objects is a functor, thus our map should both consider the image of objects and morphisms. ob $\prod(X)$ is points of X, $\forall x \in X,\ \exists U_i \in \mathscr{O}\ s.t.\ x \in U_i$. So it is natural to define

$$\Phi_{\text{ob}}(x) := \phi_i(x),\ \ for\ x \in U_i$$

It is well-defined by the closed of intersection in $\mathscr{O}$. For $x \in U_i \cap U_j$, we have two inclusions $\iota_i : U_i \hookrightarrow U_i \cap U_j$ and $\iota_j : U_i \hookrightarrow U_i \cap U_j$ implies $\phi_i(x) = \phi_{i\cap j}(x) = \phi_j(x)$, so it is independent of the choice of U_i.
The morphism map is somehow similar, notice that mor($\prod(X)$) are the homotopy class $[f] : x \longrightarrow y$ of the path. As $f([0,1])$ is compact and $\mathscr{O}$ covers X, by Lebesgue's Covering Lemma we have a subdivision $0 = t_0 < t_1 < ... < t_n = 1$ and $\{U_{i1}, U_{i2}, ..., U_{in}\} \subseteq \mathscr{O}$ such that $f([t_{k-1}, t_k]) \subset U_{i_k}$. We get a homotopy subclass on U_{i_k} by restricting our f on the interval $[t_{k-1}, t_k]$ and we get f_k such that $[f] = *_{k=1}^{n}[f_k]$ where f_k is a path in U_{i_k} form x_{k-1} to x_k. After those set up we can define the morphism map of our $\Phi : \prod(X) \longrightarrow G$

$$\Phi_{\text{mor}}[f] := \phi_{i_n}([f_n]) \circ \phi_{i_{n-1}}([f_{n-1}]) \circ ... \circ \phi_{i_2}([f_2]) \circ \phi_{i_1}([f_1])$$

It is independent of the choice of U_i again by the closed intersection of our cover thus it is well-defined. Now we can verify the universal property of the given functor which is the uniqueness of given Φ. Suppose there is another functor $\Psi : \prod(X) \longrightarrow G$ such that $\phi_i = \Psi \circ \iota_i$ for every i.
Consider any $x \in X$, there is $x \in U_i$ for some $U_i \in \mathscr{O}$ which gives the induced inclusion map $\iota_i(x) = x$. So

$$\Psi(x) = \Psi(\iota_i(x)) = \phi_i(x) = \Phi(x)$$

which says the two functor have same image on ob($\prod(X)$). For any $[f] \in \text{Hom}_{\prod(X)}(x, y)$, we have the same subdivision $[f] = *_{k=1}^{n}[f_k]$, thus $\Psi([f]) = \Psi([f_1] * [f_2] * ... * [f_n]) = \Psi([f_n]) \circ ... \circ \Psi([f_2]) \circ \Psi([f_1])$. Notice that $[f_k]$ in $\text{Hom}_{\prod(U_{i_k})}$ so it is the same after applying the functor induced by the inclusion map, that is $[f_k] = \iota_{i_k}([f_k]) \in \text{Hom}_{\prod(X)}$. In conclusion we have

$$\begin{aligned}
\Psi([f]) &= \Psi([f_n]) \circ ... \circ \Psi([f_2]) \circ \Psi([f_1]) \\
&= \Psi(\iota_{i_n}[f_n]) \circ ... \circ \Psi(\iota_{i_2}[f_2]) \circ \Psi(\iota_{i_1}[f_1]) \\
&= \phi_{i_n}([f_n]) \circ ... \circ \phi_{i_2}([f_2]) \circ \phi_{i_1}([f_1])
\end{aligned}$$

which is exactly $\Phi([f])$. Thus the universal property is verified. □

The detaily proof above is not hard to understand, the key to verify the universal property for categorical objects is to construct a proper one based on what we already have. For those categorical expert one should really easy to follow the proof just by understand four equations of the definitions and verifications. The diagram below review our proof and picture what we are working on so far.

$$
\begin{tikzcd}
 & x \arrow[rrrr, "{[f]}"] \arrow[dddl, "\Phi"'] & & & & y \arrow[dddr, "\Phi"] & \\
\prod(U_x) & & & \prod(X) & & & \prod(U_y) \\
x_1 \arrow[uur, "\iota_x"] \arrow[rr, "{[f_x]}"] \arrow[d, "\phi_x"] & & x_2 \arrow[uul, "\iota_x"'] \arrow[dr, "\phi_x"] & & y_1 \arrow[uur, "\iota_y"] \arrow[rr, "{[f_y]}"] \arrow[dl, "\phi_y"'] & & y_2 \arrow[uul, "\iota_y"'] \arrow[d, "\phi_y"] \\
\Phi(x)=\phi_x(x) \arrow[rrr, "\phi_x({[f_x]})"] \arrow[rrrrrr, bend right, "\Phi({[f]})=\Phi({[f_x]}*{[f_y]})=\phi_y({[f_y]})\circ\phi_x({[f_x]})"'] & & & \circ \arrow[rrr, "\phi_y({[f_y]})"] & & & \Phi(y)=\phi_y(y) \\
 & & & G & & &
\end{tikzcd}
$$

Proof by veriying the universal property is not hard but our proof is somehow different because our working is in the category of groupiod **GP**. Every groupiod is a category so the morphism between any two is a functor, which means we have to verify the image both on objects and morphisms in specific groupiod. If we move to the fundamental group version of van Kampen then will be more trivial.

Theorem 4.3 *(Categorical van Kampen) The group $\pi_1(X, x)$ is the colimit of the $\mathscr{O}$-shaped diagram restricted on the cover $\pi_1|_{\mathscr{O}} : \mathscr{O} \longrightarrow$ **Grp**, in mathematical words:*

$$\pi_1(X, x) \cong \mathrm{colim}_{U_i \in \mathscr{O}} \pi_1(U_i, x)$$

Proof (Categorical functor proof) The proof follows formally from our previous one but there is some details we need to talk about. Our goal is the same that to verify the universal property, but this time we focus on the category of group **Grp**. Recall we consider group as special groupiod with a single object and the elements of group as the morphisms. We consider the localize functor:

$$
\begin{aligned}
&B_{x_0} : \mathbf{GP} \longrightarrow \mathbf{Grp} \\
&\forall G \in \mathrm{ob}(\mathbf{GP}), B_{x_0}(G) = \mathrm{Hom}_G(x_0, x_0) \\
&\forall \mathcal{F} \in \mathrm{Hom}_{\mathbf{GP}}(G, H), B_{x_0}(\mathcal{F}) = \mathcal{F}|_{\mathrm{Hom}_G}(x_0, x_0)
\end{aligned}
$$

The definition for this functor on morphism is a restriction of $\mathcal{F}$ on the groupiod G which itself a category, that is, $\mathcal{F}|_{\mathrm{Hom}_G}(x_0, x_0) : \mathrm{Hom}_G(x_0, x_0) \longrightarrow \mathrm{Hom}_H(\mathcal{F}(x_0), \mathcal{F}(x_0))$. By this functor for any elements in our $\mathscr{O}$-shaped diagram, we have $B_{x_0}(\prod(U_i)) = \mathrm{Hom}_{\prod(U_i)}(x_0, x_0) = \pi(U_i, x_0)$, the same to $\prod(X)$. We have a bridge from **GP** and reconsider the last proof, the diagram below commutes

$$
\begin{tikzcd}
\mathscr{O} \arrow[r, "\prod"] & \mathbf{GP} \arrow[r, "B_{x_0}"] & \mathbf{Grp} \\
U_i \arrow[r, "\prod(*)"] \arrow[d, hook, "\iota"] & \prod(U_i) \arrow[r, "B_{x_0}(*)"] \arrow[d, "\prod(\iota)"] & \pi_1(U_i, x_0) \arrow[d, "B_{x_0}(\prod(\iota))"] \\
U_j \arrow[r, "\prod(*)"] & \prod(U_j) \arrow[r, "B_{x_0}(*)"] & \pi_1(U_j, x_0)
\end{tikzcd}
$$

In last proof we show $\prod(X) \cong \mathrm{colim}_{U_i \in \mathscr{O}} \prod(U_i)$ in **GP**, now if our functor B_{x_0} preserves the colimits in **GP** then it is done. In mathematical, we want to show

$$G = B_{x_0}(\mathrm{colim}_{U_i \in \mathscr{O}} \prod(U_i)) \cong \mathrm{colim}_{U_i \in \mathscr{O}} B_{x_0}(\prod(U_i)) = H$$

where $B_{x_0}(\mathrm{colim}_{U_i\in\mathscr{O}}\prod(U_i)) = \pi_1(X,x_0)$ and $\mathrm{colim}_{U_i\in\mathscr{O}}B_{x_0}(\prod(U_i)) = \mathrm{colim}_{U_i\in\mathscr{O}}\pi_1(X,x_0)$, for convenience we use G and H to represent those two groups. Consider the mapping $\theta : H \longrightarrow G$, for any element in H is a loop $[\gamma] \in U_i$ for some $U_i \in \mathscr{O}$. By the inclusion functor J_i from **Grp** to **GP**, $J_i([\gamma]) \in \mathrm{Hom}_{\prod(U_i)}(x_0, x_0)$, then send to $\square$

Proof (Categorical equivalence proof) Now we give another proof totally based on the power of category. For **Gp** and **Grp** there are always two natural functors. One is the inclusion of categories $\mathcal{J} : \pi_1(X,x) \longrightarrow \prod(X)$, which sends a group as one single object $*$ to a specific basepoint x, and sends the homotopy class of loops $[\gamma] \in \mathrm{Hom}_{\pi_1(X,x)}(*,*)$ to the automorphism of x in $\prod(X)$. As a dual there is a contract functor $\mathcal{F} : \prod(X) \longrightarrow \pi_1(X,x)$ with a set up that $\forall y \in X$ there exists a chosen path $\alpha_y : x \longrightarrow y$ such that $\alpha_x = c_x$ is constant path, *i.e.* $f(t) = x$, $\forall t \in [0,1]$. Moreover, for $y \in U_i$ for some $U_i \in \mathscr{O}$ lies entirely in U_i. These two functors are called categorical equivalence functors and each of them is the inverse equivalence to the other, which one should easy to verify $\mathcal{F} \circ \mathcal{J} = \mathrm{Id}_{\mathbf{Grp}}$.
We shall assume our cover $\mathscr{O}$ is finite and then general to infinite case. And the idea of proof is simple as the commutative diagram shows below, which we use $\mathcal{F}$ and $\mathcal{J}$ to build up a bridge between two colimts.

$$\begin{tikzcd}
 & \pi_1(U_1,x_0) \arrow[dddd, "\pi_1(\prod(\hookrightarrow))"'] \arrow[ddr, hook, "i_1"] \arrow[ddrrr, "\psi_1"] & & & \\
\prod(U_1) \arrow[ur, "\mathcal{F}_{U_1}"] \arrow[dd, hook, "\prod(\hookrightarrow)"'] \arrow[drrr, hook, "\iota_1"] \arrow[drrrr, "\phi_1"] & & & & \\
 & & \pi_1(X,x_0) \arrow[r, "\mathcal{J}"] & \prod(X) \arrow[r, "\exists!\Phi"] & G \\
\prod(U_2) \arrow[dr, "\mathcal{F}_{U_2}"'] \arrow[urrr, hook, "\iota_2"] \arrow[urrrr, "\phi_2"] & & & & \\
 & \pi_1(U_2,x_0) \arrow[uur, hook, "i_2"] \arrow[uurrr, "\psi_2"'] & & &
\end{tikzcd}$$

For any $y \in U_i$ by our set up to pick path $x \longrightarrow y$ entirely in U_i, thus the functors travels three categories

$$\prod(U_i) \xrightarrow{F_{U_i}} \pi_1(U_i,x_0) \xrightarrow{\psi_i} G$$

is an $\mathscr{O}$-shaped diagram $\psi_i \circ F_{U_i} : \prod|\mathscr{O} \longrightarrow \mathbf{GP}$, notice that a group could also be considered as groupiod. By the groupiod version of van Kampen, there exists a unique map in category of **GP** that $\Phi : \prod(X) \longrightarrow G$ such that $\psi_i \circ \mathcal{F}_{U_i} = \Phi \circ \iota_i$ for all i. By the uniqueness of Φ and the bridge $\mathcal{J} : \pi(X,x_0) \longrightarrow \prod(X)$, we have a unique homomorphism $\Psi = \Phi \circ \mathcal{J}$ as required, which satisfies the universal property of colimits that $\Psi \circ i_i = \psi_i$, where i_i is the inclusion from $\pi_1(U_i,x_0)$ to $\pi(X,x_0)$ induced be the inclusion as sets, and ψ_i is the map $\pi_1(U_i,x_0)$ to G similar to ϕ_i before in **GP**.
Our work in not done yet, we have to general to infinite case. For a infinite cover $\mathscr{O}$ we have $\mathscr{F}$ as the set of those finite subsets in $\mathscr{O}$ which closed under finite intersection. For specific subset $S \in \mathscr{F}$, we denote the union of $U_i \in S$ as U_S and it is clear that S is a cover of U_S. Moreover the $\mathscr{F}$ is again a finite cover can be considered as a category of $\mathscr{O}$ in finite case, so we have

$$\mathrm{colim}_{U_i\in S}\pi_1(U_i,x_0) \cong \pi_1(U_S,x_0)$$
$$\mathrm{colim}_{S_i\in\mathscr{F}}\pi_1(U_{S_i},x_0) \cong \pi_1(X,x_0)$$

Now for any infinite cover $\mathscr{O}$ we subdivide to $\mathscr{F}$ and by arguement before any loops in X has image in some U_S, so all we need to prove is

$$\mathrm{colim}_{U_i \in \mathscr{O}} \pi_1(U_i, x_0) \cong \mathrm{colim}_{S_i \in \mathscr{F}} \pi_1(U_{S_i}, x_0) \cong \mathrm{colim}_{S \in \mathscr{F}} \mathrm{colim}_{U_i \in S} \pi_1(U_i, x_0)$$

The iterated colimt is isomorphic to the single colimit $\mathrm{colim}_{(U_i,S_i)\in(\mathscr{O},\mathscr{F})}\pi_1(U_i, x_0)$. The category $(\mathscr{O}, \mathscr{F})$ has morphism $(U_1, S_1) \longrightarrow (U_2, S_2)$ if both $U_1 \subset U_2$ and $U_{S_1} \subset U_{S_2}$. So there is a natural inclusion functor that $\mathcal{I} : \mathscr{O} \longrightarrow (\mathscr{O}, \mathscr{F})$ where the diagram below commutes

$$\begin{tikzcd}
 & U_1 \arrow[r, "\mathcal{I}"] \arrow[dd, hook] & (U_1, \{U_1\}) \arrow[dd, hook] & \\
\mathscr{O} & & & (\mathscr{O}, \mathscr{F}) \\
 & U_2 \arrow[r, "\mathcal{I}"] & (U_2, \{U_2\}) &
\end{tikzcd}$$

where $\mathcal{I}$ sends U_i to its singleton set $\{U_i\}$ in $\mathscr{F}$. One should easily verify the only difference between $\mathscr{O}$ and $(\mathscr{O}, \mathscr{F})$ is that for a homomorphisms $\pi_1(U_1, x_0) \longrightarrow \pi_1(U_2, x_0)$, it only applys in $\mathscr{O}$ once but many times in $(\mathscr{O}, \mathscr{F})$ with the same result comes out, there is no new contribution to our colimit. On the other side is more trivial, by projection gives us the function $\mathcal{P} : (\mathscr{O}, \mathscr{F}) \longrightarrow \mathscr{O}$. Those functors composite with $\pi_1 : \mathscr{O} \longrightarrow \mathbf{Grp}$ gives us the isomorphism

$$\mathrm{colim}_{U_i \in \mathscr{O}} \pi_1(U_i, x_0) \cong \mathrm{colim}_{(U_i,S_i)\in(\mathscr{O},\mathscr{F})}\pi_1(U_i, x_0)$$

$$\begin{tikzcd}
 & \pi_1(U_1, x_0) \arrow[ddl, "\iota_1"'] \arrow[dddd, hook] & & & \pi_1(U_1, \{U_1\}) \arrow[ddr, "(\iota_1, I_1)"] \arrow[dddd, hook] & \\
 & & U_1 \arrow[ul, "\pi_1"'] \arrow[r, shift left, "\mathcal{I}"] \arrow[dd, hook] & (U_1, \{U_1\}) \arrow[l, shift left, "\mathcal{P}"] \arrow[ur, "\pi_1"] \arrow[dd, hook] & & \\
\mathrm{colim}_{U_i \in \mathscr{O}} \pi_1(U_i, x_0) \arrow[rrrrr, shift left] & & & & & \mathrm{colim}_{(U_i,S_i)\in(\mathscr{O},\mathscr{F})}\pi_1(U_i, x_0) \arrow[lllll, shift left] \\
 & & U_2 \arrow[dl, "\pi_1"] \arrow[r, shift left, "\mathcal{I}"] & (U_2, \{U_2\}) \arrow[l, shift left, "\mathcal{P}"] \arrow[dr, "\pi_1"] & & \\
 & \pi_1(U_2, x_0) \arrow[uul, "\iota_2"] & & & \pi_1(U_2, \{U_2\}) \arrow[uur, "(\iota_2, I_2)"'] &
\end{tikzcd}$$

The commutative diagram shows clearly how two easy functors between $\mathscr{O}$ and $(\mathscr{O}, \mathscr{F})$ build up the isomorphism. And our prove end up with

$$\mathrm{colim}_{U_i \in \mathscr{O}} \pi_1(U_i, x_0) \cong \mathrm{colim}_{(U_i,S_i)\in(\mathscr{O},\mathscr{F})}\pi_1(U_i, x_0) \cong \mathrm{colim}_{S_i \in \mathscr{F}} \pi_1(U_{S_i}, x_0) \cong \pi_1(X, x_0) \qquad \square$$

What's so intersting is that to be reversed, recall that $\mathcal{F} : \prod(X) \longrightarrow \pi_1(X, x_0)$. If we have uniquely $\Psi : \pi_1(X, x_0) \longrightarrow G$ restricts to ψ_i on each $\pi_1(U_i, x_0)$, then $\Psi \circ \mathcal{F} : \prod(X) \longrightarrow G$ restricts to $\psi_i \circ \mathcal{F}_{U_i}$ which is exactly ϕ_i, meaning it is the way to prove groupiod version if we can prove group version constructivly, the van Kampen in groupiod and groups are equivalent.

4.3 Cofibrations

For a inclusion $i : A \hookrightarrow X$ of spaces, we say it has the homotopy extension property for a space Y if every homotopy $H : A \times [0,1] \longrightarrow Y$ and for every map $f : X \longrightarrow Y$ with $f(i(a)) = H(a, 0)$ for every $a \in A$, there is a homotopy $\hat{H} : X \times [0,1] \longrightarrow Y$ such that $\hat{H}(i(a), t) = H(a, t)$ and $\hat{H}(x, 0) = f(x)$ for all $a \in A, x \in X, t \in [0,1]$

Definition 4.3 *(HEP) An inclusion $i : A \hookrightarrow X$ has a homotopy extension property if for any space Y, i has the left lifting property which makes the diagram commutes*

$$\begin{tikzcd} A \arrow[r, "H"] \arrow[d, hook, "i"'] & Y^{[0,1]} \arrow[d, "\mathrm{ev}_0"] \\ X \arrow[r, "f"] \arrow[ur, dashed, "\exists \tilde{H}"] & Y \end{tikzcd}$$

5 Covering Space

5.1 Covering Space

We have already talked about the fundamental group and fundamental groupiod for topological space, then comes up with van Kampen theorem to calculate the fundamental structure for some nontrivial space. What we more focus on is the equivalent description to van Kampen theorem, show us the power of categorical language also give us some intuition to homotopy theory.

Now we come to covering space, recall how we prove $\pi_1(S^1) = \mathbb{Z}$ by covering the real line $\mathbb{R}$ circle by circle to S^1. But covering space is more than that to compute fundamental group. Unlike any other textbook, we will see covering space in categorical language and in homotopic way.

Definition 5.1 *(Evenly Covering Space) An evenly covering space for space X consists a pair $(\hat{X}, p)$ such that $p : \hat{X} \longrightarrow X$ is surjective and $\forall x \in X$, there is an open neighborhood U_x where*

$$p^{-1}(U_x) = \bigsqcup_{i \in I} V_i$$

for some disjoint open set V_i in $\hat{X}$, and $p|_{V_i} : V_i \longrightarrow U_x$ is homeomorphism for every $i \in I$. We call $\{V_i\}$ is the sheets of $\hat{X}$ over X and $F_x = p^{-1}(x) \in \bigsqcup_{i \in I} V_i$ is a fiber of the covering p.

It should be mentioned that the difference between evenly covering space and covering space is that the preimage of any neighborhood can be written as disjoint union or not. But in Algebric Topological the covering we describe are almost all evenly, so in later words we consider cover is an evenly cover. One should verify any homomorphism is a cover. Most classical example is the covering we used to show $\pi_1(S^1) = \mathbb{Z}$ by $p = \mathbb{R} \longrightarrow S^1$. We will see the strong connection between covering space and the fundamental groupiod or group, the following theorem describes how p acts on the space.

Theorem 5.1 *(Unique path lifting theorem) For a cover $p : \hat{X} \longrightarrow X$ with $e_0 \in F_{x_0}$ where $x_0 \in X$, we have*

$$\forall f \in \{f \text{ } a \text{ } path : I \longrightarrow X | f(0) = x_0\},$$
$$\exists! g \in \{g \text{ } a \text{ } path : I \longrightarrow \hat{X} | g(0) = e_0\}$$

such that $p \circ g = f$

Proof For a evenly covered base space X, if f lies entirely on the sheet $S_{x_0} = \{V_i\}$ with respect to U_{x_0}, then $g = q \circ f$ where q is the homomorphic inverse of p such that $q|_{V_i} \circ p|_{V_i} = id|V_i$, $\forall V_i \in S_{x_0}$, and the uniqueness comes from the unique inverse for homeomorphism. For a general case, by Lebesgue's Covering Lemma I is compact that we can partition I by $0 = t_0 < t_1 < ... < t_n = 1$ according to the covering, making f maps every closed $[t_i, t_{t+1}]$ to an evenly covered neighborhood of $f(t_i)$, then by the evenly cover case and induction we can obatin the unique lifting. $\square$

What the unique path lifting theorem means is that for the cover $p : \hat{X} \longrightarrow X$, any fundamental points x_0 in based space X lifted into the fiber F_{x_0}, and $\forall e_i \in F_{x_0}$. This theorem give us the intuition that the lifting also act on the homotopic calss of each path.

Corollary 5.1 *For a covering $p : \hat{X} \longrightarrow X$ and a homotopy in X that $h : f \simeq f'$ start from x_0 lifts uniquely to a homotopy $H : g \simeq g'$ in $\hat{X}$ start from e_0.*

Proof For the homotopy $h : I \times I \longrightarrow X$, by Lebesgue's Covering Lemma we can subdivide the compact $I \times I$ into many subsquares each of which maps to a fundamental neighborhood of f, by **Theorem 5.1** we see any paths in the homotopic class lifts to a unique path that h lifts uniquely to a homotopy $H : I \times I \longrightarrow \hat{X}$ where f and f' lift to g and g' with $H : g \simeq g'$ that $H(0,0) = e_0$, and one should be easily verified that $h = p \circ H$. □

Proposition 5.1 *The induced group homomorphism $p_* : \pi_1(\hat{X}, e_i) \longrightarrow \pi_1(X, x_0)$ is a monomorphism, for all $e_i \in F_{x_0}$.*

Before discussing the algebric structure of covering space, we first define two important covering space according to special group structures.

Definition 5.2 *Consider the covering $p : \hat{X} \longrightarrow X$. We say p is regular if $\pi_1(X, x_0)/p_*(\pi_1(\hat{X}, e_i))$ is still a group, p is universal if $\pi_1(X, x_0)/p_*(\pi_1(\hat{X}, e_i)) = \pi_1(X, x_0)$ and it is the normal subgroup of $\mathbb{Z}$ so it's regular.*

Example 5.1 *Any integer multiple winding of S^1 is a normal cover, say if a covering wrap the S^1 three times it is easy to calculate $p_*(\hat{X}, e_i) = 3\mathbb{Z}$. Moreover, the covering $p : \mathbb{R} \longrightarrow S^1$ is universal because $\mathbb{R}$ is simply connected that $\pi_1(\mathbb{R}, e_i) = 0$, we also call $\mathbb{R}$ is the universal cover of S^1.*

The relationship between covering space $\hat{X}$ and based space X is defined on the structure of the underlying fundamental groups. But we want to study the structure more specifically, says we should generalize the idea of covering from topological space to any algebric objects. One way to do this is to define the covering space on categories. Once we have the categorical covering, the induced homomorphism p_* will become a covering contain more information rather than a mapping. The way to generalize the covering space is working on the category called morphism category.

Definition 5.3 *(Morphisms Category) Let $\mathscr{C}$ be any category and a $x \in \mathrm{ob}(\mathscr{C})$, the morphism category of $\mathscr{C}$ under x is a category $x\backslash\mathscr{C}$ such that*

$$\mathrm{Ob}(x\backslash\mathscr{C}) := \bigcup_{y_i \in \mathrm{Ob}(\mathscr{C})} \mathrm{Hom}_{\mathscr{C}}(x, y_i)$$

which is all the morphisms in $\mathscr{C}$ that from x to others. For any $f, g \in \mathrm{Ob}(x\backslash\mathscr{C})$, $f : x \longrightarrow y_1,\ g : x \longrightarrow .y_2$ for some $y_1, y_2 \in \mathrm{Ob}(\mathscr{C})$, then the morphism in $x\backslash\mathscr{C}$ is defined as the compostion

$$\mathrm{Hom}_{x\backslash\mathscr{C}}(f, g) := \mathrm{Hom}_{\mathscr{C}}(y_1, y_2)$$

which means the morphisms $\gamma : f \longrightarrow g$ are the morphisms $\gamma : y_1 \longrightarrow y_2$ such that $\gamma(f) := \gamma \circ f = g$ makes the diagram below commutes

$$\begin{tikzcd}
 & x \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, phantom, "\xrightarrow[\gamma\in\mathrm{Hom}_{x\backslash\mathscr{C}}(f,g)]{}"] & \\
y_1 \arrow[rr, "\gamma\in\mathrm{Hom}_{\mathscr{C}}(y_1,y_2)"'] & & y_2
\end{tikzcd}$$

The motivation of defining this is that we want to generalize our covering idea to fundamental group or groupiod, as you can see both of them can be considered as categorical objects, so the covering should be a functor.

Definition 5.4 *(Covering of Groupiods) Let $\mathscr{B}$ and $\mathscr{C}$ be two small connected category, meaning the objects form a set not a proper class, and any two object have at least one invertalbe morphism between them. Then the covering between $\mathscr{B}$ and $\mathscr{C}$ is a surjective functor $p : \mathscr{B} \longrightarrow \mathscr{C}$ such that*

$$p : \mathrm{Ob}(b\backslash\mathscr{B}) \longrightarrow \mathrm{Ob}(p(b)\backslash\mathscr{C})$$

is a bijection $\forall b \in \mathscr{B}$. Similar to the covering of topological space, for a $c \in \mathscr{C}$, the fiber of c is the set $F_c = \{b \in \mathscr{B} | p(b) = c\}$ and $\forall c \in \mathscr{C}$ we have

$$p^{-1}(\mathrm{Ob}(c\backslash\mathscr{C})) = \bigsqcup_{b_i \in F_c} \mathrm{Ob}(b_i\backslash\mathscr{B})$$

The objects in $x\backslash\mathscr{C}$ is a set of morphisms of $\mathscr{C}$ with source x we denote is as $\mathrm{St}_{\mathscr{C}}(x)$, so the covering restricts to the bijection $p : \mathrm{St}_{\mathscr{B}}(b) \longrightarrow \mathrm{St}_{\mathscr{C}}(p(b))$ and $p^{-1}(\mathrm{St}_{\mathscr{C}}(c)) = \bigsqcup_{b_i \in F_c} \mathrm{St}_{\mathscr{B}}(b_i)$.

The definition of covering of groupiods is strictly followed by the covering of topological space, with the based point x_0 changed to the morphism with source x_0. We have the statement below naturally.

Proposition 5.2 *The induced functor $\prod(p) : \prod(\hat{X}) \longrightarrow \prod(X)$ is a covering of groupiod if $p : \hat{X} \longrightarrow X$ is a covering of topological space.*

Proof This is actually the same statement of **Theorem 5.1** and **Corollary 5.1** as long as consider any path is a invertible morphism in the groupiod as a category. $\square$

The key of algebric topology is to use algebric structure to study and verfy difference topological space, and our main work is also trying to discover any potential example to build a bridge between difference mathematical field. We will see how covering space and Galois theory connected to each other by the underlying group of topological space through covering space. Now we will study more on the algebric side of covering space.

Definition 5.5 *(Automorphism group of groupiod) Let $\mathscr{C}$ be any groupiod, the automorphism of $x \in \mathscr{C}$ are all the objects in $x\backslash\mathscr{C}$ which from x sends to x itself and denoted as $\pi(\mathscr{C}, x)$, in mathematical words*

$$\pi(\mathscr{C}, x) := \mathrm{Hom}_{\mathscr{C}}(x, x)$$

It is obvious that if the $\mathscr{C}$ is any fundamental groupiod of a topological space then the automorphism group is actually the fundamental group with respect to the chosen x_0, that is

$$\pi(\prod(X), x_0) = \mathrm{Hom}_{\prod(X)}(x_0, x_0) = \pi_1(X, x_0)$$

So for any covering space $p : \hat{X} \longrightarrow X$ we have the covering of underlying groupiod with the connected of groups, the covering of groupiod indicate some information on groups.

Proposition 5.3 *Let $p : \hat{X} \longrightarrow X$ be a covering of topological space, then the induced morphism $p_* : \pi(\prod(\hat{X}), e_i) \longrightarrow \pi(\prod(X), x_0)$ is a monomorphism, where $x_0 \in X$ is the based point and $e_i \in F_{x_0} = \{e_i\}_{i \in I}$. Moreover, $\forall e_j, e_k \in F_{x_0}$, $p_*(\pi(\prod(\hat{X}), e_j))$ is conjugate to $p_*(\pi(\prod(\hat{X}), e_k))$ in $\pi(\prod(X), x_0)$.*

Proof The injectivity is trivial by the bijection of $\prod(p)$ on $\mathrm{St}(e_i)$. $\prod(\hat{X})$ is a groupiod so there is a path $g : e_j \longrightarrow e_k$, the conjugation is given by the $p(g) \in \pi(\prod(X), x_0)$, that is

$$p_*(\pi(\prod(\hat{X}), e_k)) = [p_*(g)] \circ [p_*(\pi(\prod(\hat{X}), e_j))] \circ [p_*(g)]^{-1} \qquad \square$$

The proposition is saying the underlying group of based space at x_0 is hiden in any copy of group at $e_i \in F_{x_0}$, when the covering space covers to X by p, all the group structure on F_{x_0} will onto the one group at x_0.

Proposition 5.4 *The group $p_*(\pi(\prod(\hat{X}), e_j))$ runs through all conjugates of $p_*(\pi(\prod(\hat{X}), e_i)) \in \pi(\prod(X), x_0)$ as e_j runs through F_{x_0}.*

Proof This is trivial that the surjectivity of the groupiod covering functor $\prod(p)$ on $\mathrm{St}(e_i)$ and apply the same method in **Proof 5.3** $\square$

Now the connection between covering space of topological space and groupiod, together with the underlying fundamental group is very clear as the commutative diagram shows below, we consider the fundamental group of a space as the automorphism group under the groupiod category at specific object, all our work is trying to build a categorical eyesight of algebric topology.

$$\begin{tikzcd}
\hat{X} \arrow[r, "\prod"] \arrow[d, "p"] & \prod(\hat{X}) \arrow[r, "\pi(\mathscr{C}, x)"] \arrow[d, "\prod(p)"] & \pi(\prod(\hat{X}), e_i) \arrow[r, equal] \arrow[d, "p_*"] & \pi_1(\hat{X}, e_i) \arrow[d, "p^*"] \\
X \arrow[r, "\text{underlying groupoid}"'] & \prod(X) \arrow[r, "\text{automorphism group}"'] & \pi(\prod(X), x_0) \arrow[r, equal] & \pi_1(X, x_0)
\end{tikzcd}$$

It is clear that the fundamental group of covering space should have a same corresponding relation as groupiod, if we consider the group as the automorphism of the groupiod category under the covering, then there should be some algebric relation between fundamental groups.

5.2 Covering space and Galois theory

Galois theroy is one of the most powerful algebric theory. Consider the quadratic equation $ax^2 + bx + c = 0$, we say $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is the formula of solution, namely the radical. For any higher n degree polynomial equation $\sum_{i=0}^{n} a_i x^i = 0$, the radical to this equation is a family of functions $x_i = f_i(a_0, a_1, ...a_n)$ which satisfies the equations, where any functions only required the operations of $+, -, \times, \div, \sqrt{\cdot}$. The biggest contribution of Galois Theory is the proof of the non-existence of the radical for any equations more than degree 5. In this chapter we will introduce some basic techique of Galois theory and see how it is connected to covering space.

The genius idea of Galois is to connect every equations to its symmetric group. More precisly, every equations's spliting field has a Galois groups and the existence of radical is related to the Galois group is solvable or not.

Definition 5.6 *(Ring and Field) A ring R is a set with binary operation $+$ and $\times$ where $(R, +)$ is a commutative group and $\times$ is associative and distributive over addition. A field F is a ring such that $(F \backslash \{0\}, \times)$ is a commutative group.*

The definition is not new to us, fields play very provital role in Galois theory, where Galois theory build a bridge between field extension and group theory.

Definition 5.7 *(Field Extension) Let F be any field, a field extension of F is a field L with a injective inclusion homomorphism $i : F \hookrightarrow L$. We denote the field extension as L/F. We can consider the extension field L as a vector space over F, and the dimension of L over F is called the degree of the extension, denoted as $[L : F]$.*

Example 5.2 *The complex field $\mathbb{C}$ is a field extension of real field $\mathbb{R}$ with the inclusion $i : \mathbb{R} \hookrightarrow \mathbb{C}$, and the degree of this extension is 2 because $\{1, i\}$ form a basis of $\mathbb{C}$ over $\mathbb{R}$.*

Example 5.3 *The field $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} | a, b \in \mathbb{Q}\}$ is a field extension of rational field $\mathbb{Q}$ with inclusion $i : \mathbb{Q} \hookrightarrow \mathbb{Q}(\sqrt{2})$, and the degree of this extension is also 2 because $\{1, \sqrt{2}\}$ form a basis of $\mathbb{Q}(\sqrt{2})$.*

Definition 5.8 *(spliting field) Let F be any field and $f(x) \in F[x]$ be any polynomial over F, a splitting field of $f(x)$ over F is a field extension L/F such that*

$$f(x) = a(x - \alpha_1)(x - \alpha_2)...(x - \alpha_n)$$

in $L[x]$ where $\alpha_i \in L$ are all the roots of $f(x)$ in L, and $L = F(\alpha_1, \alpha_2, ..., \alpha_n)$ is generated by adjoining all the roots to F. Morevore, such L is the minimal field with this property, meaning if there is another field extension K/F such that $f(x)$ splits in $K[x]$, then $L \subseteq K$.

Consider the polynomial $f(x) = x^2 - 2$ over $\mathbb{Q}$, the roots of $f(x)$ are $\pm\sqrt{2} \notin \mathbb{Q}$ which is not what we want. Our primry goal is to find a minimal field that contains $f(x)$'s parameters also with roots, so that we can study $f(x)$ by studying the structure of this minimal field.

Example 5.4 *Consider the polynomial $f(x) = x^2 + 1$ over $\mathbb{R}$, then the splitting field of $f(x)$ is $\mathbb{R}(i)$ because $f(x) = (x - i)(x + i)$ in $\mathbb{R}(i)[x]$ and $\mathbb{R}(i, -i) = \mathbb{R}(i) = \{a + bi|\ a, b \in \mathbb{R}\}$ is generated by adjoining the roots to $\mathbb{R}$, where i and $-i$ contribute the same extension.*

Example 5.5 *Consider the polynomial $f(x) = x^3 - 2$ over $\mathbb{Q}$, then the splitting field of $f(x)$ is $\mathbb{Q}(\sqrt[3]{2}, \omega)$ where $\omega = e^{2\pi i/3}$ because $f(x) = (x - \sqrt[3]{2})(x - \omega\sqrt[3]{2})(x - \omega^2\sqrt[3]{2})$ in $\mathbb{Q}(\sqrt[3]{2}, \omega)[x]$ and $\mathbb{Q}(\sqrt[3]{2}, \omega) = \mathbb{Q}(\sqrt[3]{2}, \omega\sqrt[3]{2}, \omega^2\sqrt[3]{2})$.*

If you have good mathematical intuition, you may find the permutation of the roots play very important role in the structure of the splitting field, and this is exactly what Galois theory is trying to do, build a bridge between field extension and group theory by studying the permutation of the roots.

Before we introduce Galois group and Galois extension, we first need to introduce normal extension and separable extension.

Definition 5.9 *(Normal Extension) Let L/F be any field extension, we say L/F is a normal extension if L is the splitting field of some polynomial over F.*

Example 5.6 *Consider the polynomial $f(x) = x^2 + 1 \in \mathbb{R}[x]$ has splitting field $\mathbb{R}(i)$, then the extension $\mathbb{R}(i)/\mathbb{R}$ is a normal extension.*

Example 5.7 *Consider the polynomial $f(x) = x^3 - 2 \in \mathbb{Q}[x]$ has splitting field $\mathbb{Q}(\sqrt[3]{2}, \omega)$, then the extension $\mathbb{Q}(\sqrt[3]{2}, \omega)/\mathbb{Q}$ is a normal extension.*

Example 5.8 *The extension $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is* ***not normal****. Although $\sqrt[3]{2}$ is a root of $f(x) = x^3 - 2 \in \mathbb{Q}[x]$, the polynomial does not split completely in $\mathbb{Q}(\sqrt[3]{2})$ since the other two roots $\omega\sqrt[3]{2}$ and $\omega^2\sqrt[3]{2}$ are not contained in this field ($\mathbb{Q}(\sqrt[3]{2}) \subset \mathbb{R}$ but $\omega \notin \mathbb{R}$). Thus, $\mathbb{Q}(\sqrt[3]{2})$ is not the splitting field of any polynomial over $\mathbb{Q}$.*

Definition 5.10 *(Separable Extension) Let L/F be any field extension, we say L/F is a separable extension if every element $\alpha \in L$ is the root of some separable polynomial over F, that is a polynomial whose all roots are distinct in its splitting field.*

Example 5.9 *Consider the polynomial $f(x) = x^2 + 1 \in \mathbb{R}[x]$ has splitting field $\mathbb{R}(i)$, then the extension $\mathbb{R}(i)/\mathbb{R}$ is a separable extension because $f(x)$ has two distinct roots $\pm i$ in $\mathbb{R}(i)$.*

Example 5.10 *Let k be a field of characteristic $p > 0$, and let $F = k(t)$ be the field of rational functions over k. Consider the polynomial $f(x) = x^p - t \in F[x]$. This polynomial is irreducible over F but is* ***not separable****. In any extension field containing a root α of $f(x)$, we have $\alpha^p = t$, and thus*

$$f(x) = x^p - t = x^p - \alpha^p = (x - \alpha)^p$$

so α is the only root with multiplicity p. The extension $F(\alpha)/F$ is a ***non-separable*** *(purely inseparable) extension.*

Now we can introduce Galois group and Galois extension.

Definition 5.11 *(Galois Group) Let L/F be any field extension, the Galois group of L/F is the group of all field automorphisms of L that fix every element of F, denoted as $\mathrm{Gal}(L/F)$, that is*

$$\mathrm{Gal}(L/F) := \{\sigma \in \mathrm{Aut}(L) | \sigma(a) = a, \forall a \in F\}$$

Example 5.11 *Consider $f(x) = x^2+1 \in R[x]$ has roots in the extension field $\mathbb{R}(i)$, then to find the Galois group $\mathrm{Gal}(\mathbb{R}(i)/\mathbb{R}) = \{\sigma \in \mathrm{Aut}(\mathbb{R}(i)) | \sigma(a) = a, \forall a \in \mathbb{R}\}$, it is easy to see that $\sigma(i)^2 = \sigma(i^2) = \sigma(-1) = -1$ because the filed automorphism is first a ring homomorphism, thus $\sigma(i) = \pm i$. So*

$$\mathrm{Gal}(\mathbb{R}(i)/\mathbb{R}) = \{id, \sigma\} \cong \mathbb{Z}/2\mathbb{Z}$$

where $\mathrm{id}(a + bi) = a + bi$ and $\sigma(a + bi) = a - bi$ for all $a, b \in \mathbb{R}$.

Example 5.12 *Consider $f(x) = x^3 - 2 \in \mathbb{Q}[x]$ has roots in the extension field $\mathbb{Q}(\sqrt[3]{2}, \omega)$, then to find the Galois group $\mathrm{Gal}(\mathbb{Q}(\sqrt[3]{2}, \omega)/\mathbb{Q}) = \{\sigma \in \mathrm{Aut}(\mathbb{Q}(\sqrt[3]{2}, \omega)) | \sigma(a) = a, \forall a \in \mathbb{Q}\}$, we first see that $\sigma(\sqrt[3]{2})$ is also a root of $f(x)$ so $\sigma(\sqrt[3]{2}) = \omega^k \sqrt[3]{2}$ for some $k = 0, 1, 2$. Similarly, $\sigma(\omega)$ is also a root of $x^2 + x + 1$ so $\sigma(\omega) = \omega$ or ω^2. Thus we have six different automorphisms given by the choice of $\sigma(\sqrt[3]{2})$ and $\sigma(\omega)$, and it is easy to verify these six automorphisms form a group isomorphic to S_3.*

Next we will introduce Galois extension and see how Galois group act on the extension field.

Definition 5.12 *(Galois Extension) Let L/F be any field extension, we say L/F is a Galois extension if $|\mathrm{Gal}(L/F)| = [L : F]$.*

Proposition 5.5 *A Galois extension is both normal and separable.*

Proof Let L/F be a Galois extension. By definition, L/F is finite and $|\mathrm{Gal}(L/F)| = [L : F]$. We first show that L/F is separable. Let $\alpha \in L$ and let $m(x) \in F[x]$ be its minimal polynomial. Consider the orbit of α under the action of $G = \mathrm{Gal}(L/F)$:

$$\{\sigma(\alpha) : \sigma \in G\}$$

Let $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_r$ be the distinct elements in this orbit. Every $\sigma \in G$ permutes the α_i, so the polynomial

$$f(x) = \prod_{i=1}^{r} (x - \alpha_i)$$

has coefficients fixed by all $\sigma \in G$. Since L/F is Galois, the fixed field of G is exactly F, so $f(x) \in F[x]$. Since $m(x)$ is the minimal polynomial of α over F, we have $m(\alpha) = 0$, and for any $\sigma \in G$, $m(\sigma(\alpha)) = \sigma(m(\alpha)) = 0$. Thus all α_i are roots of $m(x)$, so $f(x)$ divides $m(x)$ in $L[x]$, hence also in $F[x]$ by Gauss's Lemma. But $m(x)$ is irreducible over F, so $m(x) = f(x)$ up to a constant factor. Therefore $m(x)$ has distinct roots $\alpha_1, \ldots, \alpha_r$, so α is separable over F. Since α was arbitrary, L/F is separable.

Then we show L/F is a normal extension. Let $f(x) \in F[x]$ be an irreducible polynomial that has a root $\alpha \in L$. We need to show that $f(x)$ splits completely in L. Consider the minimal polynomial $m(x)$ of α over F. Since $f(x)$ is irreducible and has α as a root, $f(x) = m(x)$ up to a constant factor. From Step 1, we know $m(x)$ splits into distinct linear factors in some extension.

Let β be any root of $m(x)$ in an algebraic closure. Then there exists an F-isomorphism $\sigma : F(\alpha) \to F(\beta)$ with $\sigma(\alpha) = \beta$. Since L/F is separable and L is the splitting field of some polynomial over $F(\alpha)$, the isomorphism σ extends to an automorphism $\tilde{\sigma} \in \mathrm{Gal}(L/F)$.

Therefore $\beta = \tilde{\sigma}(\alpha) \in L$, so all roots of $m(x)$ lie in L. Hence $m(x)$ splits completely in $L[x]$, and thus $f(x)$ splits completely in $L[x]$. This shows that L/F is normal. $\square$

Example 5.13 *Consider the polynomial* $f(x) = x^2 + 1 \in \mathbb{R}[x]$ *has splitting field* $\mathbb{R}(i)$*, then the Galois group* $\mathrm{Gal}(\mathbb{R}(i)/\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$ *has order* 2 *and the degree of the extension* $[\mathbb{R}(i) : \mathbb{R}] = 2$*, so* $\mathbb{R}(i)/\mathbb{R}$ *is a Galois extension.*

Theorem 5.2 *(Characterization of Galois Extension) Let* L/F *be any field extension, then the following statements are equivalent:*

1. L/F *is a Galois extension.*
2. F *is the fixed field of* $\mathrm{Gal}(L/F)$*, that is* $F = \{a \in L | \sigma(a) = a, \forall \sigma \in \mathrm{Gal}(L/F)\}$.
3. L *is a splitting field of a separable polynomial over* F.

Proof We prove the equivalences cyclically: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1).
(1) $\Rightarrow$ (2): Assume L/F is Galois. Let $G = \mathrm{Gal}(L/F)$ and define the fixed field

$$L^G = \{a \in L \mid \sigma(a) = a \text{ for all } \sigma \in G\}$$

Clearly $F \subseteq L^G$. Suppose for contradiction that $F \subsetneq L^G$.
Then there exists $\alpha \in L^G \backslash F$ with minimal polynomial $m(x) \in F[x]$ of degree > 1. Since L/F is separable, $m(x)$ has distinct roots. Since L/F is normal, all roots of $m(x)$ lie in L.
Let $\beta \neq \alpha$ be another root of $m(x)$. There exists an F-embedding $\sigma : F(\alpha) \to L$ with $\sigma(\alpha) = \beta$. This extends to an F-automorphism $\tilde{\sigma} \in G$ because L/F is normal.
But $\alpha \in L^G$ implies $\tilde{\sigma}(\alpha) = \alpha$, contradicting $\tilde{\sigma}(\alpha) = \beta \neq \alpha$. Therefore $L^G = F$.
(2) $\Rightarrow$ (3): Assume $F = L^{\mathrm{Gal}(L/F)}$. Since L/F is finite, take an F-basis $\{\alpha_1, \dots, \alpha_n\}$ of L. Let $G = \mathrm{Gal}(L/F) = \{\sigma_1, \dots, \sigma_n\}$. Consider the polynomial

$$f(x) = \prod_{i=1}^{n} \prod_{j=1}^{n} (x - \sigma_i(\alpha_j))$$

The coefficients of $f(x)$ are symmetric polynomials in the $\sigma_i(\alpha_j)$, and are fixed by all $\sigma \in G$ since applying σ just permutes the factors. By assumption (2), these coefficients lie in F. Thus $f(x) \in F[x]$.
Since each α_j appears among the roots $\sigma_i(\alpha_j)$ (take $\sigma_1 = \mathrm{id}$), L is generated over F by roots of $f(x)$. Also, $f(x)$ is separable because its roots are distinct (they are images of basis elements under automorphisms). Therefore L is the splitting field of the separable polynomial $f(x)$ over F.
(3) $\Rightarrow$ (1): Assume L is the splitting field of a separable polynomial $f(x) \in F[x]$. Then L/F is normal by definition of splitting field, and separable because $f(x)$ is separable. Since L/F is finite, normal, and separable, it is Galois. □

The Galois extension build a strong connection between field extension and its Galois group, and the next step is to see how Galois group act on the extension field.

Theorem 5.3 *(Fundamental Theorem of Galois Theory) Let* L/F *be a Galois extension with Galois group* $G = \mathrm{Gal}(L/F)$*, then there is a one-to-one correspondence between the set of intermediate fields* K *with* $F \subseteq K \subseteq L$ *and the set of subgroups* H *of* G*, given by the maps*

$$K \longmapsto \mathrm{Gal}(L/K) = \{\sigma \in G | \sigma(a) = a, \forall a \in K\}$$
$$H \longmapsto L^H = \{a \in L | \sigma(a) = a, \forall \sigma \in H\}$$

Furthermore, the following properties hold:

1. K *is a Galois extension of* F *if and only if* H *is a normal subgroup of* G*, in which case* $\mathrm{Gal}(K/F) \cong G/H$.
2. $[L : K] = |H|$ *and* $[K : F] = [G : H]$.

Proof We divide the proof into several parts. First we show the maps are well-defined and inverses. Let K be an intermediate field with $F \subseteq K \subseteq L$. Since L/F is Galois, L/K is also Galois (as it is the splitting field of the same separable polynomial over K). Thus $\mathrm{Gal}(L/K)$ is a subgroup of G. Let H be a subgroup of G. The fixed field L^H is clearly an intermediate field with $F \subseteq L^H \subseteq L$. We now show these maps are inverses. For any intermediate field K:

$$L^{\mathrm{Gal}(L/K)} = K$$

This holds because L/K is Galois, so the fixed field of $\mathrm{Gal}(L/K)$ is exactly K.
For any subgroup $H \leq G$:

$$\mathrm{Gal}(L/L^H) = H$$

To see this, let $H' = \mathrm{Gal}(L/L^H)$. Clearly $H \subseteq H'$ since every $\sigma \in H$ fixes L^H by definition.
Now consider the degrees:

$$[L : L^H] = |H'| \quad \text{(since } L/L^H \text{ is Galois)}$$
$$[L : L^H] \leq |H| \quad \text{(by Artin's lemma)}$$

Thus $|H'| \leq |H|$, but since $H \subseteq H'$, we must have $H = H'$. Morevore, for any corresponding pair $K = L^H$ and $H = \mathrm{Gal}(L/K)$:

$$[L : K] = |\mathrm{Gal}(L/K)| = |H|$$
$$[K : F] = \frac{[L : F]}{[L : K]} = \frac{|G|}{|H|} = [G : H]$$

which proves property (2).
Suppose $H \lhd G$ is normal and $K = L^H$. We show K/F is Galois. For any $\sigma \in G$ and $a \in K$, and any $\tau \in H$, we have: $\tau(\sigma(a)) = \sigma(\sigma^{-1}\tau\sigma(a)) = \sigma(a)$, since $\sigma^{-1}\tau\sigma \in H$ by normality, and thus fixes $a \in K = L^H$. Therefore $\sigma(a) \in L^H = K$, so $\sigma(K) = K$. This allows us to define a restriction map:

$$\rho : G \to \mathrm{Gal}(K/F), \quad \sigma \mapsto \sigma|_K$$

The kernel of ρ is exactly $\mathrm{Gal}(L/K) = H$, so we get an injection: $G/H \hookrightarrow \mathrm{Gal}(K/F)$. Comparing orders:

$$[G : H] = [K : F] \leq |\mathrm{Gal}(K/F)| \leq [K : F]$$

Thus equality holds throughout, so K/F is Galois with $\mathrm{Gal}(K/F) \cong G/H$. Conversely, if K/F is Galois, then for any $\sigma \in G$ and $\tau \in H = \mathrm{Gal}(L/K)$, and any $a \in K$ we have $\sigma^{-1}\tau\sigma(a) = \sigma^{-1}(\sigma(a)) = a$, since τ fixes $\sigma(a) \in K$ (as K/F is normal). Thus $\sigma^{-1}\tau\sigma \in H$, so $H \lhd G$. □

Example 5.14 *Consider the polynomial $f(x) = x^2 + 1 \in \mathbb{R}[x]$ has splitting field $\mathbb{R}(i)$, then the Galois group $\mathrm{Gal}(\mathbb{R}(i)/\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$ has only two subgroups: the trivial group $\{id\}$ and the whole group itself. By the fundamental theorem of Galois theory, there are only two intermediate fields: $\mathbb{R}$ corresponding to $\mathrm{Gal}(\mathbb{R}(i)/\mathbb{R})$ and $\mathbb{R}(i)$ corresponding to the trivial group.*

Example 5.15 *Consider the polynomial $f(x) = x^3 - 2 \in \mathbb{Q}[x]$ has splitting field $\mathbb{Q}(\sqrt[3]{2}, \omega)$, then the Galois group $\mathrm{Gal}(\mathbb{Q}(\sqrt[3]{2}, \omega)/\mathbb{Q}) \cong S_3$ has several subgroups, for example, the subgroup $H = \{id, (123), (132)\} \cong \mathbb{Z}/3\mathbb{Z}$ is a normal subgroup of S_3. By the fundamental theorem of Galois theory, there is an intermediate field K corresponding to H, and $K/\mathbb{Q}$ is a Galois extension with Galois group $\mathrm{Gal}(K/\mathbb{Q}) \cong S_3/H \cong \mathbb{Z}/2\mathbb{Z}$.*

The fundamental theorem of Galois theory build a one-to-one correspondence between the intermediate field and the subgroup of Galois group, which is very similar to the connection between covering space and the underlying fundamental group. This is not a coincidence, in fact, there is a deep connection between covering space and Galois theory through the underlying fundamental group of topological space.

Theorem 5.4 *(Covering space and Galois correspondence) Let X be a connected, locally path-connected, and semi-locally simply connected topological space with base point $x_0 \in X$. Then there is a one-to-one correspondence between the set of equivalence classes of connected covering spaces of X (up to isomorphism over X) and the set of conjugacy classes of subgroups of the fundamental group $\pi_1(X, x_0)$. In mathematical words ,*

$$\{\textit{Connected covering spaces of } X\}/\cong \quad \longleftrightarrow \quad \{\textit{Conjugacy classes of subgroups of } \pi_1(X, x_0)\}$$

Furthermore, for a connected covering space $p : \hat{X} \longrightarrow X$ corresponding to a subgroup $H \leq \pi_1(X, x_0)$, the following properties hold:

1. *The covering space $p : \hat{X} \longrightarrow X$ is a Galois (regular) covering if and only if H is a normal subgroup of $\pi_1(X, x_0)$, in which case the group of deck transformations $\mathrm{Deck}(\hat{X}/X)$ is isomorphic to the quotient group $\pi_1(X, x_0)/H$.*

2. *The number of sheets of the covering space $p : \hat{X} \longrightarrow X$ is equal to the index $[\pi_1(X, x_0) : H]$.*

6 Basic homotopy theory

In previous chapters, we have introduced categorical language, and see how category plays important role in algebraic topology. In Algebric topology we always focus on difference functors, trying to find some topological invariant that can distinguish different topological spaces. Homotopy theory is one of the most important branch of algebraic topology, which study topological spaces up to homotopy equivalence. In this chapter we will introduce some basic concepts of homotopy theory.

Definition 6.1 *(Homotopy) Let X and Y be two topological spaces, two continuous maps $f, g : X \longrightarrow Y$ are said to be homotopic if there exists a continuous map $H : X \times I \longrightarrow Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$, where $I = [0, 1]$ is the unit interval. The map H is called a homotopy between f and g.*

Definition 6.2 *(Homotopy Equivalence) Let X and Y be two topological spaces, we say X and Y are homotopy equivalent if there exist continuous maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ such that $g \circ f$ is homotopic to the identity map id_X and $f \circ g$ is homotopic to the identity map id_Y. In this case, the maps f and g are called homotopy equivalences.*

Example 6.1 *Consider the unit circle $S^1 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$ and the punctured plane $\mathbb{R}^2 \backslash \{0\}$, then the map $f : S^1 \longrightarrow \mathbb{R}^2 \backslash \{0\}$ defined by $f(x, y) = (x, y)$ and the map $g : \mathbb{R}^2 \backslash \{0\} \longrightarrow S^1$ defined by $g(x, y) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right)$ are homotopy equivalences, since $g \circ f = \mathrm{id}_{S^1}$ and $f \circ g$ is homotopic to $\mathrm{id}_{\mathbb{R}^2 \backslash \{0\}}$ via the homotopy*

$$H : (\mathbb{R}^2 \backslash \{0\}) \times I \longrightarrow \mathbb{R}^2 \backslash \{0\}, \quad H((x, y), t) = \left((1 - t)x + t\frac{x}{\sqrt{x^2 + y^2}}, (1 - t)y + t\frac{y}{\sqrt{x^2 + y^2}} \right)$$

Homotopy equivalence is an equivalence relation on the category of topological spaces, and we can form the homotopy category **hTop** by taking topological spaces as objects and homotopy classes of continuous maps as morphisms. Homotopy equivalence is a weaker notion than homeomorphism, since homeomorphic spaces are always homotopy equivalent, but the converse is not true in general. For example, the unit circle S^1 and the punctured plane $\mathbb{R}^2 \backslash \{0\}$ are homotopy equivalent but not homeomorphic.

Our fundamental group and fundamental groupiods are defined on the homomorpy class of one-dimension, recall that

$$\pi_1(X, x_0) = \mathrm{Hom}_{\prod(X)(x_0, x_0)} = \pi(\prod(X), x_0) \tag{1}$$

which we consider the foundamental group as the automorphism group of the fundamental groupiod as a category at object x_0. Geometrically, all the automorphism form the loops structures at x_0 which is S^1 up to homotopy, so what we can conclude is that $\pi_1(X, x_0)$ actually shows how we can map S^1 into X based at x_0 up to homotopy. This is a very important observation, and actually is the geometrical definition of fundamental group.

π_1 helps us to detect whether a one-dimensional 'rubber band ' cen be contracted to a point in the space X of not. If $\pi_1(X, x_0)$ is not trivial, then there exists at least one one-dimensional hole in our space near x_0.

So what we will do next is obvious, we will generalize the fundamental group to higher dimension, that is, we want to study how we can map S^n into X based at x_0 up to homotopy for any $n \geq 1$. This leads us to the definition of higher homotopy groups, which helps us to detect higher dimensional holes in the space X near x_0 just by simply studying the structure of these groups.

Definition 6.3 *(n-cube and its boundary) Let $I = [0, 1]$ be the unit interval. For any $n \in \mathbb{N} \setminus \{0\}$, the n-cube is defined as:*

$$I^n = \underbrace{I \times I \times \cdots \times I}_{n \text{ times}} = \{(t_1, t_2, \ldots, t_n) \mid t_i \in I,\ \forall 1 \leq i \leq n\} \subset \mathbb{R}^n$$

The boundary of I^n is:

$$\partial I^n = \cup_{i=1}^{n} \{(t_1, t_2, \ldots, t_n) \in I^n \mid t_i \in \{0, 1\}\}$$

So the n-th homotopy group is the structures of all mapping from $n-th$ cube to the topological space X with the boundary ∂I^n mapped to the base point x_0 up to homotopy. By the identification of the boundary ∂I^n to a point, we can see that this is equivalent to study all the mapping from S^n to X based at x_0 up to homotopy, in mathematical words, $I^n/\partial I^n \cong S^n$.

Definition 6.4 *(Higher homotopy groups) Let (X, x_0) be a pointed topological space, for any $n \in \mathbb{N} \setminus \{0\}$, the n-th homotopy group of (X, x_0) is defined as*

$$\pi_n(X, x_0) = \{f : (I^n, \partial I^n) \longrightarrow (X, x_0)\}/\sim$$

where $\sim$ is the homotopy relation between continuous maps that fix the boundary ∂I^n pointwise. The group operation is given by concatenation of maps along one coordinate direction.

$$\begin{tikzcd}
(I^n, \partial I^n) \arrow[dr, "p_1"] \arrow[drr, "\pi_n(X,x_0)/\sim"] & & \\
& (I^n/\partial I^n, \partial I^n/\partial I^n) \arrow[r] \arrow[dl] & (X, x_0) \\
(S^n, s_0) \arrow[urr, "\pi_n(X,x_0)/\sim"'] & &
\end{tikzcd}$$