Final Year Project Template at School of Mathematics & Physics

The Homotopic Interpretation of Mathematical Proofs: a Categorical and Type-Theoretic Study of Structural Equivalence

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By

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To My Mom and Dad

Abstract

This final year project aims to explore the profound structural unity within pure mathematics. We introduce **category theory** as the core framework, arguing that it provides a unifying language that transcends specific computations and is grounded in ”relationships” and ”structures”, capable of penetrating the surface differences between fields such as algebra and topology to reveal their inherent connections.

This claim is concretely demonstrated through two key case studies: First, we revisit the **van-Kampen Theorem via categorical colimits**, proving that its two formulations – in terms of the fundamental group and the fundamental groupoid – are equivalent statements and also equivalent to classical version. This replaces traditional combinatorical constructions with purely structural arguments. Second, we systematically establish the **categorical equivalence** between Galois theory and classification theorem of covering spaces through **antitone Galois connections**, revealing that **Galois theory and classification of covering spaces are equivalent as they are the applications of Galois connections in algebra and topology**. This naturally leads to Grothendieck’s theory of the **étale fundamental group**, elevating the unity to a higher level.

Building on this work, the thesis further proposes a more fundamental philosophical and mathematical assertion: viewing ’**propositions as types**’ and ’**proofs as morphisms or paths**’. From this perspective, the equivalence between proofs becomes a **higher-order homotopy**, making it possible to construct a new foundation for mathematics based on topology and homotopy theory. As a feasibility demonstration of this vision, we present a **preliminary formalization** of the fundamental groupoid within the framework of **Homotopy Type Theory (HoTT)** at the end of the thesis. This not only marks a step from categorical interpretations to type-theoretic formalizations but also opens the door to further explorations of the homotopic nature of mathematical proofs, taking a critical step toward our ultimate understanding of the identity of mathematical structures.

Keywords: Category Theory, Galois Connections, Galois Theory, Classification Theorem of Covering Space, Homotopy Theory, Univalent Foundations

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Chapter 2

Category Theory

Any readers who are familiar with category theory can skip this section directly to next part. In this section we will introduce some basic concepts in category theory that are necessary for understanding the rest of this paper.
For a systematically category theory study, check [4, Basic Category Theory]. And basically all categorical notation we will use later will follow it.

2.1 Category

Definition 2.1.1 (Category) A **category** $\mathscr{C}$ consists of a collection $ob(\mathscr{C})$ of **objects**, and $\forall A, B \in ob(\mathscr{C})$, there is a collection $\mathscr{C}(A, B)$ of **morphisms** from A to B. With following three axioms satisfied:
1. **Compostion**: $\forall A, B, C \in ob(\mathscr{C})$, if $f \in \mathscr{C}(A, B)$ *and* $g \in \mathscr{C}(B, C)$, then there is a function $g \circ f \in \mathscr{C}(A, C)$ called compostion of f and g.
2. **Associativity**: $(h \circ g) \circ f = h \circ (g \circ f),\ \forall f \in \mathscr{C}(A, B),\ g \in \mathscr{C}(B, C),\ h \in \mathscr{C}(C, D)$.
3. **Identity laws**: $\forall A \in ob(\mathscr{C}),\ \exists 1_A \in \mathscr{C}(A, A)$, called the identity of A. And $\forall f \in \mathscr{C}(A, B)$, we have $f \circ 1_A = f = 1_B \circ f$.

Example 2.1.2 (Category of Sets: **Set**) For a category $\mathscr{C}$, where $ob(\mathscr{C})$ is a collection of sets, here we consider each set as one object, no matter the cardinality of it. Given two set A and $B \in ob(\mathscr{C})$, the mapping or morphism between two sets is exactly a function from A to B. Together with all the sets we have and the functions between each two sets we call them **Set**, the category of set.

Example 2.1.3 (Category of Groups and Rings: **Grp** and **Ring**) We have a collection of groups, and a morphism between every two given group G and H which is so-called group homomorphism. Then all these groups together with the group homomorphisms are called category **Grp** of groups. Similarly, there is a category **Ring** of rings and ring homomorphisms.

Example 2.1.4 (Category of Vector Spaces over field k: $\mathbf{Vect}_k$) For a field k, $\mathbf{Vect}_k$ consists the vector fields over k and the mapping between two vector spaces H to W which will be the k linear transformations from H to W, i.e. $\mathcal{L}(H, W)$.

Example 2.1.5 (Category of Topological Spaces: **Top**) There is a collection of topological spaces and the mapping between topological spaces are continous maps. Together the topological spaces and maps are called **Top**.

Example 2.1.6 (Category of nothing: $\emptyset$) There is a collection of nothing and no morphisms between nothing, these called empty category $\emptyset$.

Example 2.1.7 (Category of one object: **1**) There is a category **1** with only one object in the collection and only Identity map.

Example 2.1.8 (Discrete Category) A category $\mathscr{C}$ is discrete if $\forall A, B \in ob(\mathscr{C})$ and $A \neq B$, $\mathrm{Hom}_{\mathscr{C}}(A, B) = \emptyset$. This does not mean there is no mapping in $\mathscr{C}$, notice that $1_A \in \mathrm{Hom}_{\mathscr{C}}(A, A),\ \forall A$.

Example 2.1.9 (One object category constructed by a group) A group actually is a one object category. Differ this from category **1** that **1** has only identity but for group one element in it is a morphism. Let us put in a clear way. We have a group $G = \{e, g_1, g_2, ...\}$, consider a category $\mathscr{C}$ that $\mathrm{Ob}_{\mathscr{C}} = G$, the identity morphism in $\mathscr{C}(G, G)$ is actually $e \in G$:

$$e(G) = e \cdot G = \{e \cdot e, e \cdot g_1, e \cdot g_2 ...\} = \{e, g_1, g_2, ...\} = G$$

and $\forall g \in G$, $g(G) \subseteq G$ from the closure of group structure, if $|G| < \infty$, then $g(G) = G,\ \forall g \in G$. The corresponding table below helps you to understand the isomorphism between mathematical structure.

Category $\mathscr{C}$ with single object A	**Group G**
Maps in $\mathscr{C}$	Elements in G
$\circ$ in $\mathscr{C}$	$\cdot$ in G
$1_A \in \mathscr{C}(A, A)$	$e_G \in G$

We remark that the category of one mathematical object is a collection of some structural objects not necessarily all the objects, and we provide a example say one object category of group.
Now we put our focus into the morphisms in category, given A and B as object of category $\mathscr{C}$, the mapping in $\mathscr{C}(A, B)$ should not necessarily be so-called functions or transformations, we name the morphisms as transformations it is because for $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$, we have $f : A \longrightarrow B$, gives us feeling that the morphism f trans A into B.
We should consider it more abstract, the type of $f(A) = B$ is actually a one directional relation of A and B. If f is not some machine but a not comprehensive statement, for example: $f :=$ 'is bigger than', then $f(A) = B$ is a full statement:

$$f(A) = B \Longleftrightarrow A \longrightarrow B \Longleftrightarrow A \textit{ is bigger than } B$$

Consider mapping as relation between different objects is one of core idea in category theory, it is a great abstraction and acrroding to this we can find many isomorphism between different mathematical structures.

Definition 2.1.10 (Opposite Category) a category noted $\mathscr{C}^{op}$ is said to be the opposite or dual category of given category $\mathscr{C}$, it has exactly the same object with all the arrows in $\mathscr{C}$ reversed, thas is:

$$ob(\mathscr{C}^{op}) = ob(\mathscr{C}) \ \ \textit{and} \ \ \mathscr{C}^{op}(B, A) = \mathscr{C}(A, B)$$

The 'Category' is not a unique object but a structure of mathematical objects. For instance, Group is something in a set satisfy associativity and there is an identity and inverce under specific operation, the set in definition of group can be anything, they could be functions, person, computer program, formula for Rubik's cube... So similarly, here the category of some mathematical object is not represent all these object in the collection. You can construct any category you want with few objects as long as you give the morphisms and they satisfy the axioms.

Take **Set** as example, the category of set not necessarily contain all sets, if you construct a set contains limited number of sets and follow the axioms to be a category, it still called **Set**.

You may have an intuition that if we not restrict the size of $\mathrm{Hom}_{\mathscr{C}}$ for specific $\mathscr{C}$, we have a trouble with Russell's Paradox because we are talking about "set of sets". To avoid such hindrance, we introduce the concept of small category.

Definition 2.1.11 (Locally Small Category) We say a category $\mathscr{C}$ is locally small if $\forall A, B \in \mathrm{Ob}_{\mathscr{C}}$, $\mathrm{Hom}_{\mathscr{C}}(A, B)$ is a set.

Example 2.1.12 The category **Set** is locally small because for any two sets $A, B \in ob(\mathbf{Set})$, the morphism set $\mathrm{Hom}_{\mathbf{Set}}(A, B)$ is the set of all functions from A to B, which is a set.

Example 2.1.13 The category **Grp** is locally small because for any two groups $G, H \in ob(\mathbf{Grp})$, the morphism set $\mathrm{Hom}_{\mathbf{Grp}}(G, H)$ is the set of all group homomorphisms from G to H, which is a set.

2.2 Functor

Definition 2.2.1 (Functor) We say a map of categories $\mathcal{F} : \mathscr{C} \longrightarrow \mathscr{D}$ is a functor, if it sends every A in $ob(\mathscr{C})$ to a $\mathcal{F}(A)$ in $ob(\mathscr{D})$ and a morphism $f : A \longrightarrow B$ of $\mathscr{C}$ to a morphism $\mathcal{F}(f) : \mathcal{F}(A) \longrightarrow \mathcal{F}(B)$ of $\mathscr{D}$, while satisfies two axioms that

$$\mathcal{F}(id_A) = id_{\mathcal{F}(A)} \quad and \quad \mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$$

where $A \in ob(\mathscr{C})$ and $f, g \in mor(\mathscr{C})$

The fundamental group map $\pi_1(*, *)$ can be considered as a functor from the category of topological space with the basepoint **Top*** to **Grp**. For any topological space X with a basepoint x specified, the functor send the pair $(X, x) \in ob(\mathbf{Top^*})$ to its fundamental group $\pi_1(X, x) \in \mathbf{Grp}$, with give us the algebric structure of the loops start at basepoint x in space X. In this case any $f \in \mathbf{Top^*}(X, Y)$ not only a continous map from X to Y but also a basepoint-preserving $f : (X, x) \longrightarrow (Y, y)$, with its image under the functor $\pi_1(f) : \pi_1(X, x) \longrightarrow \pi_1(Y, y)$.

For examples apart from algebric topology, we have forgetful functors from **Grp** to **Set** which just by its name, the group forgets its structure under the operation but keep its members as a set. And free functors can be considered as dual functor of forgetful, send a set to a group with an operation and add more elements in the set to make it a group.

One type a functor is widely used in categorical language, for a locally small category $\mathscr{C}$ and $A \in ob(\mathscr{C})$, we have $H^A = \mathrm{Hom}(A, -) : \mathscr{C} \longrightarrow \mathbf{Set}$. The morphism functor send every element X in the category to the morphism set $\mathrm{Hom}(A, X)$, which is the set of all morphisms from A to X. And the morphism map under functor is $H^A(g) = \mathrm{Hom}(A, g) : \mathrm{Hom}(A, X) \longrightarrow \mathrm{Hom}(A, Y)$, simply by $f \longmapsto g \circ f$ for all $f : X \longrightarrow Y$.

But if the postion of given $A \in ob(\mathscr{C})$ switch then everything changed. For similar morphism functor $H_A = \mathrm{Hom}(-, A)$, the narrow preserving diagram make no sense, the image $H_A(g)$ cannot be defined for certain $g \in \mathrm{Hom}(X, A)$, so the functor is actually defined on the opposite category of $\mathscr{C}$, that is, $H_A = \mathrm{Hom}(-, A) : \mathscr{C} \longrightarrow \mathbf{Set}$ (See diagram below). This gives us the motivation to define the special functor on the opposite category.

$$
\begin{tikzcd}
X \arrow[r, "H^A"] \arrow[d, "f"'] & \mathrm{Hom}(X,A) \arrow[d] \\
Y \arrow[r, "H^A"] & \mathrm{Hom}(Y,A)
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "f"'] & A \\
Y \arrow[ur, "cannot\ define"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
X \arrow[r, "H^A"] & \mathrm{Hom}(X,A) \arrow[d] \\
Y \arrow[u, "f'"] \arrow[r, "H^A"] & \mathrm{Hom}(Y,A)
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "g"] & A \\
Y \arrow[u, "f'"] \arrow[ur, "g \circ f"'] &
\end{tikzcd}
$$

Definition 2.2.2 (contravariant functor) Let $\mathscr{C}$ and $\mathscr{D}$ be categories, a functor $\mathscr{C}^{op} \longrightarrow \mathscr{D}$ is said to be the contravariant functor from $\mathscr{C}$ to $\mathscr{D}$.

Having introduced functors and contravariant functors, a natural question arises: when do two functors form a "symmetric pair" that encode a reversible translation between categories? This leads to the central concept of **adjoint functors**, which is the categorical formalisation of Galois connections.

Definition 2.2.3 (Adjoint functors) Let $\mathcal{F} : \mathscr{C} \longrightarrow \mathscr{D}$ and $\mathcal{G} : \mathscr{D} \longrightarrow \mathscr{C}$ be two functors. We say $\mathcal{F}$ is **left adjoint** to $\mathcal{G}$ (and $\mathcal{G}$ is **right adjoint** to $\mathcal{F}$), written $\mathcal{F} \dashv \mathcal{G}$, if there exists a natural bijection

$$\Phi_{c,d} : \mathrm{Hom}_{\mathscr{D}}\big(\mathcal{F}(c), d\big) \xrightarrow{\simeq} \mathrm{Hom}_{\mathscr{C}}\big(c, \mathcal{G}(d)\big)$$

for every $c \in \mathscr{C}$ and $d \in \mathscr{D}$, such that for any morphisms $f : c' \to c$ in $\mathscr{C}$ and $g : d \to d'$ in $\mathscr{D}$ the following diagram commutes:

$$
\begin{tikzcd}
\mathrm{Hom}_{\mathscr{D}}\big(\mathcal{F}(c), d\big) \arrow[r, "\Phi_{c,d}"] \arrow[d, "\mathcal{F}(f)^* \circ g_*"'] & \mathrm{Hom}_{\mathscr{C}}\big(c, \mathcal{G}(d)\big) \arrow[d, "f^* \circ \mathcal{G}(g)_*"] \\
\mathrm{Hom}_{\mathscr{D}}\big(\mathcal{F}(c'), d'\big) \arrow[r, "\Phi_{c',d'}"] & \mathrm{Hom}_{\mathscr{C}}\big(c', \mathcal{G}(d')\big)
\end{tikzcd}
$$

where f^* denotes pre-composition with f and g_* denotes post-composition with g.

The definition precisely captures the idea that morphisms starting from $\mathcal{F}(c)$ in $\mathscr{D}$ correspond one-to-one to morphisms ending at $\mathcal{G}(d)$ in $\mathscr{C}$, and this correspondence respects composition in both categories. In the special case where $\mathscr{C}$ and $\mathscr{D}$ are preordered sets viewed as categories, the adjunction condition reduces exactly to the monotone Galois connection $F(a) \leq b \Longleftrightarrow a \leq G(b)$.

Example 2.2.4 (Free–forgetful adjunction) The most ubiquitous example of an adjunction is between the **free functor** $\mathcal{F} : \mathbf{Set} \longrightarrow \mathbf{Vect}_{\mathbb{R}}$ and the **forgetful functor** $\mathcal{G} : \mathbf{Vect}_{\mathbb{R}} \longrightarrow \mathbf{Set}$. Here $\mathcal{F}$ sends a set X to the free real vector space $\mathbb{R}^{(X)}$ with basis X, and $\mathcal{G}$ simply discards the vector-space structure, returning the underlying set.
The adjunction $\mathcal{F} \dashv \mathcal{G}$ is witnessed by the natural bijection

$$\mathrm{Hom}_{\mathbf{Vect}_{\mathbb{R}}}\big(\mathbb{R}^{(X)}, V\big) \ \simeq\ \mathrm{Hom}_{\mathbf{Set}}\big(X, \mathcal{G}(V)\big),$$

which states that a linear map from the free vector space on X to a vector space V is uniquely determined by its values on the basis X, i.e., by an ordinary set map $X \to \mathcal{G}(V)$. This is exactly the universal property of a free object.

Example 2.2.5 (Hom–tensor adjunction) In the category $\mathbf{Vect}_{\mathbb{R}}$, fix a vector space W. Define two functors:

$$\mathcal{F}(V) = V \otimes W, \qquad \mathcal{G}(U) = \mathrm{Hom}_{\mathbb{R}}(W, U).$$

Then $\mathcal{F} \dashv \mathcal{G}$ because there is a natural isomorphism

$$\mathrm{Hom}_{\mathbb{R}}(V \otimes W, U) \;\simeq\; \mathrm{Hom}_{\mathbb{R}}\big(V, \mathrm{Hom}_{\mathbb{R}}(W, U)\big),$$

which is the familiar "currying" operation for linear maps. This adjunction underlies many dualities in algebra and geometry.

Example 2.2.6 (Galois connection as adjunction) Let $(P, \leq)$ and $(Q, \leq)$ be two partially ordered sets, regarded as categories. A monotone Galois connection $(F : P \to Q, G : Q \to P)$ is precisely an adjunction $F \dashv G$ in this categorical setting. Indeed, the condition

$$F(p) \leq q \iff p \leq G(q)$$

is exactly the statement that the Hom-sets (which are either empty or singletons) are in bijection. Thus the theory of adjoint functors is a genuine generalisation of the order-theoretic Galois connection. We will come back to this later.

The power of the adjunction language becomes apparent when one notices that many fundamental constructions in mathematics—free objects, limits, exponentials, sheafification—are most cleanly described as left or right adjoints. In the following sections we will see how both Galois theory and the classification of covering spaces fit perfectly into this pattern, revealing a deep structural unity between algebra and topology.
Beyond covariant and contravariant functors and adjoint functors, several important variants appear naturally in mathematical practice.

Definition 2.2.7 (Faithful, full, and fully faithful functors) A functor $\mathcal{F} : \mathscr{C} \longrightarrow \mathscr{D}$ is said to be faithful if for every pair of objects $X, Y \in \mathscr{C}$, the map $\mathrm{Hom}_{\mathscr{C}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathscr{D}}(\mathcal{F}(X), \mathcal{F}(Y))$ is injective; full if the same map is surjective; and fully faithful if it is both full and faithful.

Example 2.2.8 Consider the forgetful functor $U : \mathbf{Grp} \longrightarrow \mathbf{Set}$ which sends a group to its underlying set and a group homomorphism to the corresponding set function. This functor is faithful: if two group homomorphisms $f, g : G \to H$ induce the same function on the underlying sets, then $f = g$ as homomorphisms. However, U is not full because not every set function between the underlying sets of two groups is a group homomorphism. For instance, any permutation of $\mathbb{Z}$ as a set that does not preserve addition is a morphism in **Set** but not in **Grp**.

Example 2.2.9 Let **FinSet** denote the category whose objects are all finite sets and whose morphisms are all functions between them. The inclusion functor $\iota : \mathbf{FinSet} \longrightarrow \mathbf{Set}$ sends each finite set to itself (as an object of **Set**) and each function between finite sets to the same function (as a morphism in **Set**). This functor is faithful because the inclusion of Hom-sets $\mathrm{Hom}_{\mathbf{FinSet}}(X, Y) \hookrightarrow \mathrm{Hom}_{\mathbf{Set}}(X, Y)$ is injective—indeed, it is literally the identity map on the set of functions. It is full because for any two finite sets X, Y, every set map $f : X \to Y$ is automatically a morphism in **FinSet**, so the inclusion map on Hom-sets is surjective as well. Hence ι is fully faithful: it induces a bijection $\mathrm{Hom}_{\mathbf{FinSet}}(X, Y) \cong \mathrm{Hom}_{\mathbf{Set}}(X, Y)$ for every pair of finite sets X, Y.

Example 2.2.10 Another example is the fundamental group functor $\pi_1 : \mathbf{Top}_* \longrightarrow \mathbf{Grp}$. This functor is not faithful: two different pointed maps may induce the same group homomorphism on fundamental groups. For instance, any two null-homotopic maps from $(S^1, *)$ to itself both induce the trivial homomorphism $\mathbb{Z} \to \mathbb{Z}$, yet they need not be equal as continuous maps. It is also not full:

not every group homomorphism between fundamental groups is realized by a continuous map. The full and faithful properties are therefore independent and capture different aspects of how a functor relates the structures of two categories. For example, the forgetful functor **Grp** $\longrightarrow$ **Set** is faithful but not full, as not every set map between groups is a homomorphism. The inclusion functor from the category of finite sets into all sets is fully faithful.

Another fundamental construction is the product of categories and the corresponding notion of a bifunctor.

Definition 2.2.11 (Bifunctor) Given three categories $\mathscr{C}, \mathscr{D}, \mathscr{E}$, a bifunctor is a functor

$$\mathcal{F} : \mathscr{C} \times \mathscr{D} \longrightarrow \mathscr{E},$$

where $\mathscr{C} \times \mathscr{D}$ is the product category whose objects are pairs (c, d) and morphisms are pairs (f, g) with componentwise composition.

Example 2.2.12 The most important example is the Hom-bifunctor. For a locally small category $\mathscr{C}$, we have

$$\mathrm{Hom}_{\mathscr{C}}(-,-) : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \longrightarrow \mathbf{Set},$$

which sends a pair (X, Y) to the set $\mathrm{Hom}_{\mathscr{C}}(X, Y)$, and a pair of morphisms $(f : X' \to X, g : Y \to Y')$ to the map

$$\mathrm{Hom}_{\mathscr{C}}(f, g) : \mathrm{Hom}_{\mathscr{C}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(X', Y'), \quad h \longmapsto g \circ h \circ f.$$

This bifunctor plays a crucial role in the theory of adjunctions, as the adjunction isomorphism is precisely a natural isomorphism between two such Hom-bifunctors.

2.3 Natural transformations

We have defined the categories and functors as the mapping of two categories so far, the definition of functors is actually equivalent to say the diagram below commutes.

$$\begin{tikzcd}
\mathscr{C} \arrow[d, shift right=3, "\mathcal{F}\ \mathcal{G}\ \mathcal{H}"'] \arrow[d] \arrow[d, shift left=3] \\
\mathscr{D}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & X \arrow[dl, "\mathcal{F}"'] \arrow[dr, "\mathcal{G}"] \arrow[drr, "\mathcal{H}"] \arrow[dd, "f"] & & \\
\mathcal{F}(X) \arrow[rr, dotted, no head] \arrow[dd, "\mathcal{F}(f)"] & & \mathcal{G}(X) \arrow[r, dotted, no head] \arrow[dd, "\mathcal{G}(f)"] & \mathcal{H}(X) \arrow[dd] \\
 & Y \arrow[dl, "\mathcal{F}"] \arrow[dr, "\mathcal{G}"'] \arrow[drr, "\mathcal{H}"] & & \\
\mathcal{F}(Y) \arrow[rr, dotted, no head] & & \mathcal{G}(Y) \arrow[r, dotted, no head] & \mathcal{H}(Y)
\end{tikzcd}$$

So it is so natural to define a new mapping to fill in the gaps in the dashed line at the base of the triangle. To be precise, we have to make the bottom rectangular commutes. Such mapping between two functors comes up in a natural way thus we call it natural transformations.

Definition 2.3.1 Given two categories $\mathscr{C}$ and $\mathscr{D}$, and two functors $\mathcal{F}, \mathcal{G} : \mathscr{C} \longrightarrow \mathscr{D}$, the natural transformation is a map of functors $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$, which consists a morphism $\alpha_X : \mathcal{F}(X) \longrightarrow \mathcal{G}(X)$ for all $X \in ob(\mathscr{C})$ such that for all $f : X \longrightarrow Y$, the following diagram commutes

$$\begin{tikzcd}
\mathcal{F}(X) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "\alpha_X"] & \mathcal{F}(Y) \arrow[d, "\alpha_Y"] \\
\mathcal{G}(X) \arrow[r, "\mathcal{G}(f)"] & \mathcal{G}(Y)
\end{tikzcd}$$

Example 2.3.2 Consider the category **Top** of topological spaces, and two functors $\mathcal{F}, \mathcal{G} : \textbf{Top} \longrightarrow \textbf{Top}$, where $\mathcal{F}(X) = X$ and $\mathcal{G}(X) = X \times [0,1]$ for any $X \in ob(\textbf{Top})$, and for any morphism $f : X \longrightarrow Y$, $\mathcal{F}(f) = f$ and $\mathcal{G}(f) = f \times id_{[0,1]}$. Then there is a natural transformation $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$, where for each topological space X, the morphism $\alpha_X : X \longrightarrow X \times [0,1]$ is defined by $\alpha_X(x) = (x, 0)$ for all $x \in X$. This natural transformation satisfies the commutativity condition for all morphisms in **Top**.

Example 2.3.3 Consider the category $\textbf{Vect}_{\mathbb{R}}$ of real vector spaces and linear maps. Define two functors $\mathcal{F}, \mathcal{G} : \textbf{Vect}_{\mathbb{R}} \longrightarrow \textbf{Vect}_{\mathbb{R}}$ by:

$$\mathcal{F}(V) = V \oplus V, \quad \mathcal{G}(V) = V \otimes \mathbb{R}^2,$$

where $\oplus$ denotes direct sum and $\otimes$ denotes tensor product. On a linear map $f : V \to W$, we set $\mathcal{F}(f) = f \oplus f$ and $\mathcal{G}(f) = f \otimes \mathrm{id}_{\mathbb{R}^2}$.
There is a natural transformation $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ whose component at a vector space V is the canonical isomorphism

$$\alpha_V : V \oplus V \longrightarrow V \otimes \mathbb{R}^2, \quad (v_1, v_2) \longmapsto v_1 \otimes e_1 + v_2 \otimes e_2,$$

where $\{e_1, e_2\}$ is the standard basis of $\mathbb{R}^2$. For any linear map $f : V \to W$, the diagram

$$\begin{array}{ccc} V \oplus V & \xrightarrow{f \oplus f} & W \oplus W \\ \Big\downarrow{\scriptstyle \alpha_V} & & \Big\downarrow{\scriptstyle \alpha_W} \\ V \otimes \mathbb{R}^2 & \xrightarrow{f \otimes \mathrm{id}_{\mathbb{R}^2}} & W \otimes \mathbb{R}^2 \end{array}$$

commutes because both paths send (v_1, v_2) to $f(v_1) \otimes e_1 + f(v_2) \otimes e_2$. Hence α is indeed a natural transformation.

Example 2.3.4 Let **Grp** be the category of groups and homomorphisms. Consider the abelianization functor $\mathrm{Ab} : \textbf{Grp} \longrightarrow \textbf{Ab}$, where **Ab** is the category of abelian groups, defined by

$$\mathrm{Ab}(G) = G/[G, G],$$

the quotient of G by its commutator subgroup $[G, G]$, and $\mathrm{Ab}(f)$ is the induced homomorphism on quotients.
There is a natural transformation $\eta : \mathrm{id}_{\textbf{Grp}} \longrightarrow U \circ \mathrm{Ab}$, where $U : \textbf{Ab} \longrightarrow \textbf{Grp}$ is the forgetful functor (treating an abelian group as a group). The component at a group G is the canonical projection

$$\eta_G : G \longrightarrow G/[G, G], \quad g \longmapsto g[G, G].$$

For any group homomorphism $f : G \to H$, the diagram

$$\begin{array}{ccc} G & \xrightarrow{\quad f \quad} & H \\ \Big\downarrow{\scriptstyle \eta_G} & & \Big\downarrow{\scriptstyle \eta_H} \\ G/[G, G] & \xrightarrow{\mathrm{Ab}(f)} & H/[H, H] \end{array}$$

commutes because $\mathrm{Ab}(f)(g[G, G]) = f(g)[H, H] = \eta_H(f(g))$. This natural transformation encodes the universal property of abelianization: every homomorphism from G to an abelian group factors uniquely through η_G.

A natural transformation $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ is called a *natural isomorphism* if each component $\alpha_X : \mathcal{F}(X) \to \mathcal{G}(X)$ is an isomorphism in $\mathscr{D}$. In this case we write $\mathcal{F} \cong \mathcal{G}$ and say the two functors are naturally isomorphic.

Example 2.3.5 In the category $\mathbf{FinVect}_k$ of finite-dimensional vector spaces over a field k, the double dual functor $(-)^{**}$ is naturally isomorphic to the identity functor. The natural isomorphism $\eta : \mathrm{id} \longrightarrow (-)^{**}$ has components

$$\eta_V : V \longrightarrow V^{**}, \quad \eta_V(v)(\varphi) = \varphi(v),$$

which are isomorphisms precisely when V is finite-dimensional.

Given two categories $\mathscr{C}$ and $\mathscr{D}$, we can form the *functor category* $\mathscr{D}^{\mathscr{C}}$, whose objects are functors from $\mathscr{C}$ to $\mathscr{D}$ and whose morphisms are natural transformations between them. Composition in this category is given by vertical composition of natural transformations.

Definition 2.3.6 (Vertical composition) Given three functors $\mathcal{F}, \mathcal{G}, \mathcal{H} : \mathscr{C} \longrightarrow \mathscr{D}$ and natural transformations $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ and $\beta : \mathcal{G} \longrightarrow \mathcal{H}$, the vertical composition $\beta \circ \alpha : \mathcal{F} \longrightarrow \mathcal{H}$ is defined componentwise by $(\beta \circ \alpha)_X = \beta_X \circ \alpha_X$ for each $X \in \mathscr{C}$.

There is also a horizontal composition of natural transformations. Suppose we have functors $\mathcal{F}, \mathcal{G} : \mathscr{C} \longrightarrow \mathscr{D}$ and $\mathcal{H}, \mathcal{K} : \mathscr{D} \longrightarrow \mathscr{E}$, together with natural transformations $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ and $\beta : \mathcal{H} \longrightarrow \mathcal{K}$. Then the horizontal composite $\beta * \alpha : \mathcal{H} \circ \mathcal{F} \longrightarrow \mathcal{K} \circ \mathcal{G}$ is defined by $(\beta * \alpha)_X = \beta_{\mathcal{G}(X)} \circ \mathcal{H}(\alpha_X)$.

Proposition 2.3.7 *(Naturality as a family of commutative diagrams) A collection of morphisms* $\{\alpha_X : \mathcal{F}(X) \to \mathcal{G}(X)\}_{X \in \mathscr{C}}$ *constitutes a natural transformation* $\alpha : \mathcal{F} \to \mathcal{G}$ *if and only if for every morphism* $f : X \to Y$ *in* $\mathscr{C}$*, the diagram*

$$\begin{array}{ccc} \mathcal{F}(X) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(Y) \\ \downarrow{\scriptstyle \alpha_X} & & \downarrow{\scriptstyle \alpha_Y} \\ \mathcal{G}(X) & \xrightarrow{\mathcal{G}(f)} & \mathcal{G}(Y) \end{array}$$

commutes. In particular, a natural transformation is completely determined by its components on objects.

An important application of natural transformations is the alternative description of adjunctions. An adjunction $\mathcal{F} \dashv \mathcal{G}$ can be equivalently described by two natural transformations: the *unit* $\eta : \mathrm{id}_{\mathscr{C}} \longrightarrow \mathcal{G}\mathcal{F}$ and the *counit* $\varepsilon : \mathcal{F}\mathcal{G} \longrightarrow \mathrm{id}_{\mathscr{D}}$, satisfying the triangle identities $(\varepsilon\mathcal{F}) \circ (\mathcal{F}\eta) = \mathrm{id}_{\mathcal{F}}$ and $(\mathcal{G}\varepsilon) \circ (\eta\mathcal{G}) = \mathrm{id}_{\mathcal{G}}$. This formulation highlights that adjunctions are not merely bijections of Hom-sets, but coherent families of morphisms connecting the functors.

Example 2.3.8 For the free-forgetful adjunction $\mathcal{F} \dashv \mathcal{G}$ between **Set** and $\mathbf{Vect}_{\mathbb{R}}$, the unit $\eta_X : X \longrightarrow \mathcal{G}\mathcal{F}(X)$ sends an element $x \in X$ to the basis vector e_x in the free vector space $\mathbb{R}^{(X)}$, while the counit $\varepsilon_V : \mathcal{F}\mathcal{G}(V) \longrightarrow V$ is the linear extension map that evaluates a formal linear combination of vectors in V.

2.4 Limits and colimits

Now we introduce two of the most important and powerful concepts in category theory. Limits and colimits give us a way to discover the mathematical structure universally and uniquely. Both of them are defined after a special functor.

Definition 2.4.1 (Category-shaped diagram) Let $\mathscr{C}$ and $\mathscr{D}$ be category and small category. The functor $\mathcal{F} : \mathscr{D} \longrightarrow \mathscr{C}$ is a $\mathscr{D}$-shaped diagram in $\mathscr{C}$.

We have the category $\mathscr{D}[\mathscr{C}]$ called $\mathscr{D}$-shaped diagram category in $\mathscr{C}$ where $\mathrm{Hom}(\mathcal{F}, \mathcal{F}')$ are the natural transformations. And our limits will be defined in the image of one specific $\mathscr{D}$-shaped diagram in the category $\mathscr{C}$. We already know for any two $D, D' \in \mathscr{D}$, the image of $\mathcal{F}$ make the triangle commutes. Moreover, for fixed $A \in \mathscr{C}$, it should be commutative with every image of elements in $\mathscr{D}$ under the functor.

Definition 2.4.2 (Cone) A cone on the $\mathscr{D}$-shaped diagram functor $\mathcal{F}$ consists a vertex $A \in \mathrm{Ob}(\mathscr{C})$ and the family of maps in $\mathrm{Hom}_{\mathscr{C}}$, that $f_{D_i} : A \longrightarrow \mathcal{F}(D_i)$ for every $D_i \in \mathrm{Ob}(\mathscr{D})$, such that $\forall g_i : D_i \longrightarrow D_{i+1}$ in $\mathscr{D}$, the cone diagram commutes:

$$
\begin{tikzcd}
 & & & A \arrow[dlll, "f_1"'] \arrow[dll, "f_2"'] \arrow[dl, "f_3"'] \arrow[d, "f_4"] \arrow[dr, "f_5"] \arrow[drr, "f_{k-1}"] \arrow[drrr, "f_k"] & & & \\
\mathcal{F}(D_1) \arrow[r, "\mathcal{F}(g_1)"'] & \mathcal{F}(D_2) \arrow[r, "\mathcal{F}(g_2)"'] & \mathcal{F}(D_3) \arrow[r, "\mathcal{F}(g_3)"'] & \mathcal{F}(D_4) \arrow[r, "\mathcal{F}(g_4)"'] & (D_5) \arrow[r, dotted] & \mathcal{F}(D_{k-1}) \arrow[r, "\mathcal{F}(g_{k-1})"'] & \mathcal{F}(D_k)
\end{tikzcd}
$$

The cone is unnessarily to be defined on the family of all objects D_i in $\mathscr{D}$, we denote this cone as $(f_i : A \longrightarrow \mathcal{F}(D_i))_{i \in I}$ if we pick out a specific countable set I in $\mathrm{Ob}(\mathscr{C})$.

Such cone-shaped structure is easy to find in any category and the number of such cones is large. So what we want to find is a so-called best cone which can represented as the top cone of every cone and give us an universal structure of the category.

Definition 2.4.3 (Limits and colimits) The limits of $\mathcal{F}$ is a vertex of the cone (φ_i: $\mathrm{lim}\mathcal{F} \longrightarrow \mathcal{F}(D_i))_{i \in I}$ satisfying that for any cone of $\mathcal{F}$: $(f_i : A \longrightarrow \mathcal{F}(D_i))_{i \in I}$, there is a unique map in $mor(\mathscr{C})$ $\tilde{f} : A \longrightarrow \mathrm{lim}\mathcal{F}$ such that $\varphi_i \circ \tilde{f} = f_i$, $\forall i \in I$. The colimits is the dual of limits which is defined by reversing arrows. These are equivalent to say such diagram commutes for the simple case of any two $D_1, D_2 \in \mathrm{ob}(\mathscr{D})$:

$$
\begin{tikzcd}
 & \mathrm{lim}\mathcal{F} \arrow[ddl, "\varphi_1"'] \arrow[ddr, "\varphi_2"] & \\
 & A \arrow[u, "\tilde{f}"'] \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\
\mathcal{F}(D_1) \arrow[rr, "\mathcal{F}(d_1)"'] & & \mathcal{F}(D_2)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathrm{colim}\mathcal{F} \arrow[d, "\tilde{g}"] & \\
 & A & \\
\mathcal{F}(D_1) \arrow[uur, "\phi_1"] \arrow[ur, "g_1"] \arrow[rr, "\mathcal{F}(d_1)"'] & & \mathcal{F}(D_2) \arrow[uul, "\phi_2"'] \arrow[ul, "g_2"']
\end{tikzcd}
$$

In categorical way, for one $\mathscr{D}$-shaped diagram functor $\mathcal{F}$, the image category $\mathscr{C}$ can considered as the category with objects are the vertex of cones defined on the functor $\mathcal{F}$, and $\mathrm{lim}\mathcal{F}$ is the terminal object while $\mathrm{colim}\mathcal{F}$ is the initial in such category.
The abstract definition of limits becomes concrete in familiar mathematical settings. Several special cases of limits and colimits appear ubiquitously across mathematics.

Example 2.4.4 (Products and coproducts) Let $\mathscr{D}$ be a discrete category (no non-identity morphisms). A $\mathscr{D}$-shaped diagram in $\mathscr{C}$ is simply a family of objects $\{C_i\}_{i \in I}$. A limit of such a diagram is called a *product*, denoted $\prod_{i \in I} C_i$, with projection maps $\pi_j : \prod_i C_i \to C_j$. In **Set**, the product is the Cartesian product; in **Top**, it is the product topology; in **Grp**, it is the direct product of groups.

Dually, a colimit of a discrete diagram is a *coproduct*, denoted $\coprod_{i\in I} C_i$, with inclusion maps $\iota_j : C_j \to \coprod_i C_i$. In **Set**, the coproduct is the disjoint union; in **Top**, it is the disjoint union topology; in **Grp**, it is the free product.

Example 2.4.5 (Equalizers and coequalizers) Let $\mathscr{D}$ be the category $\bullet \rightrightarrows \bullet$ with two parallel arrows. A diagram of shape $\mathscr{D}$ in $\mathscr{C}$ is a pair of morphisms $f, g : A \to B$. The limit of this diagram, called an *equalizer*, is an object E with a morphism $e : E \to A$ such that $f \circ e = g \circ e$, universal among all such pairs. In **Set**, the equalizer of $f, g : X \to Y$ is $\{x \in X \mid f(x) = g(x)\}$ with the inclusion map. The colimit of the same diagram is a *coequalizer*, given by a morphism $q : B \to Q$ with $q \circ f = q \circ g$, universal among such maps. In **Set**, the coequalizer of $f, g : X \to Y$ is the quotient of Y by the equivalence relation generated by $f(x) \sim g(x)$.

Limits and colimits are not guaranteed to exist in an arbitrary category. A category is called *complete* if it has all (small) limits, and *cocomplete* if it has all (small) colimits. Many familiar categories such as **Set**, **Top**, **Grp**, and $\mathbf{Vect}_k$ are both complete and cocomplete.

Proposition 2.4.6 *(Preservation of limits) A functor $\mathcal{F} : \mathscr{C} \longrightarrow \mathscr{D}$ is said to* preserve limits *if whenever $\{p_i : L \to C_i\}$ is a limit cone for a diagram in $\mathscr{C}$, then $\{\mathcal{F}(p_i) : \mathcal{F}(L) \to \mathcal{F}(C_i)\}$ is a limit cone for the image diagram in $\mathscr{D}$. Right adjoint functors preserve limits, and left adjoint functors preserve colimits.*

Example 2.4.7 The forgetful functor $U : \mathbf{Grp} \longrightarrow \mathbf{Set}$ is a right adjoint (to the free group functor), hence it preserves limits. Indeed, the underlying set of a product of groups is the Cartesian product of their underlying sets. Similarly, the free group functor $\mathcal{F} : \mathbf{Set} \longrightarrow \mathbf{Grp}$ is a left adjoint, so it preserves colimits: the free group on a disjoint union is the free product of the free groups on the components.

Limits and colimits provide a universal language for describing constructions that are "glued together" from simpler pieces (colimits) or that "simultaneously satisfy a family of conditions" (limits). In the next section, we will see how this universality is precisely captured by the concept of an adjoint functor, linking the abstract notion of limits to the concrete Galois connections we aim to study.

2.5 Cofibrations

For a inclusion $i : A \hookrightarrow X$ of spaces, we say it has the homotopy extension property for a space Y if every homotopy $H : A \times [0,1] \longrightarrow Y$ and for every map $f : X \longrightarrow Y$ with $f(i(a)) = H(a, 0)$ for every $a \in A$, there is a homotopy $\hat{H} : X \times [0,1] \longrightarrow Y$ such that $\hat{H}(i(a), t) = H(a, t)$ and $\hat{H}(x, 0) = f(x)$ for all $a \in A, x \in X, t \in [0,1]$

Definition 2.5.1 (HEP) An inclusion $i : A \hookrightarrow X$ has a homotopy extension property if for any space Y, i has the left lifting property which makes the diagram commutes

$$\begin{tikzcd} A \arrow[r, "H"] \arrow[d, hook, "i"'] & Y^{[0,1]} \arrow[d, "\mathrm{ev}_0"] \\ X \arrow[r, "f"] \arrow[ur, dashed, "\exists \tilde{H}"] & Y \end{tikzcd}$$

2.6 Fundamental groupoid in categorical language

Now we introduce a new mathematical construction, the groupoid. The name groupoid is come from the algebric structure group but more general. A group can consider as a one object category in the very first example in category, for a singleton $\{*\}$ we have all its automorphisms as a group, which means the category with single object $*$ a point and all the elements in group as the morphisms in category. Groupoid is defined similar to this but on more than one objects.

Definition 2.6.1 (Groupoid) A groupoid $(G, \circ)$ is a category which every morphism is isomorphism.

It is not hard to see a group is a groupoid, but the reverse is not. More clearly for $A \in \text{ob}(G)$ where $G \in \mathbf{GP}$ we have $\text{Hom}_G(A, A) = Aut(A)$ is a group and a one object category. One fact is obvious that the category $\mathbf{GP}$ is a category of categories, so every morphism is a functor.
Recall the definition of fundamental group, which is strictly dependent on the choice of base point x_0. Somehow we know for a path-connected space X we have $\pi_1(X, x_0) \cong \pi_1(X, y_0)$, but this is according to the choice of the path from x_0 to y_0, thus we still cannot say 'the fundamental group of X' but 'the fundamental group of X on the point x_0'.
This comes out as the motivation to define a new fundamental structure on the generalization of groups.

Definition 2.6.2 (Fundamental groupoid) For a topological space X and $x, y \in X$. We have $Path(x, y) := \{\gamma \in C^0([0,1], X) | \gamma(0) = x, \gamma(1) = y\}$. The Fundamental groupoid of a topological space X is a category $\prod(X)$ where $\text{ob}(X) = X$, and $\text{Hom}(x, y) = \text{Path}(x, y)/\sim$, where $\sim$ is the equivalence relation under homotopy class respect to the chosen two points.

You should verify it quickly and easily that

$$\text{Hom}_{\prod(X)(x_0, x_0)} = \pi_1(X, x_0)$$

The construction of fundamental groupoid will give us convenient way to study the structure of a topological space. Any continous map $f : X \longrightarrow Y$ induces a group homomorphism $f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, f(x_0))$ while we have to pick $f(x_0)$ properly. However the advantages of groupoids gives us a induced functor $\prod(f) : \prod(X) \longrightarrow \prod(Y)$ and this is more clean and formal. The diagram below show the difference between fundamental groups and fundamental groupoids. Both π_1 and $\prod$ are functors but groupoid induces functor (It is ok that you may think $\pi_1(f)$ is also a functor because group is one object category, but this brings us nothing new to describe the structure of space and we still have the hindrence to find the basepoint).

$$\begin{tikzcd}
\mathbf{Top}^* & \mathbf{Grp} & \{*\} \in \text{ob}_{\mathbf{Top}^*} \\
X \arrow[r, "\pi_1"] \arrow[d, "f"] & \pi_1(X, x_0) \arrow[d, "\pi_1(f)"] & x \arrow[l, "\pi_1"'] \arrow[d, "f"] \\
Y \arrow[r, "\pi_1"] & \pi_1(Y, f_0) & f(y) \arrow[l, "\pi_1"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{Top}^* & \mathbf{GP} \\
X \arrow[r, "\Pi"] \arrow[d, "f"] & \prod(X) \arrow[d, "\Pi(f)"] \\
Y \arrow[r, "\Pi"] & \prod(Y)
\end{tikzcd}$$

Chapter 3

A generalization for van Kampen Theorem

3.1 Formal van Kampen Theorem

So far we know some fundamental groups of simple topological space. The idea is that we shall consider different topological space are constructed by those simple objects and figure out a formula to compute those fundamental group from what we already know.
Before we get into the theorem, let us first image what the fundamental group of space X will be if it can be decomposed into two open sets both path-connected and contain the basepoint. Take the shape ∞ as example it can be decomposed into the union of two S^1 and the basepoint is the intersection of two S^1s. At first guess $\pi_1(\infty)$ somehow should be the product group $\mathbb{Z} \times \mathbb{Z}$ or the direct sum $\mathbb{Z} \oplus \mathbb{Z}$. But both of these is commutative, contradicts to the fact that once you get into a S^1 from basepoint you must finish the loop before you get into the other S^1. In mathematical words, if a^2ba represents go twice conter-clockwise in left S^1 then go right S^1 and left again, this path should be distinct and never be the same as a^3b. For a detailed discussion of this topological space, see [1, Section 1.2].
So in general, given a collection of open sets A_α which our topological space X can be decomposed to, we wish to construct a single group containing all $\pi_1(A_\alpha)$ as subgroups with non-commutative structure. That introduce the free product of groups.

Definition 3.1.1 (categorical free product of groups) Let $(G_i)_{i\in I}$ be a family of groups. The free product $*_{i\in I}G_i$ is a group satisfying the universal property:

$$\exists \textit{homomorphisms } \iota_j : G_j \longrightarrow *_{i\in I}G_i, \textit{ s.t.}$$
$$\textit{For any group } H \textit{ and any family of homomorphisms } f_j : G_j \longrightarrow H, \forall j \in I$$
$$\exists! \textit{homomorphism } \phi : *_{i\in I}G_i \longrightarrow H \textit{ making the diagram commute } \forall j \in I$$

$$\begin{tikzcd}
G_j \arrow[rr, "\iota_j"] \arrow[dr, "f_j"'] & & *_{i\in I}G_i \arrow[dl, "\phi"] \\
& H &
\end{tikzcd}$$

The definition is under categorical language and it may be abstract. For a formal definition we have to construct the free product of any family of groups by defining the 'words' in the product. Here is an example for an equivalent constructive definition

Definition 3.1.2 (formal free product of groups) For a family of groups $(G_i)_{i\in I}$, a word is a finite sequence $s_1 s_2 ... s_n$, where $s_k \in G_i \backslash \{e\}$ for some $i \in I$. A word is said be reduced if any two adjacent letters $s_i s_{i+1}$ belong to different groups. The free product $*_{i\in I} G_i$ is the group that elements are reduced words concatenated by the following reduction rules:
i). If adjacent letters belong different groups, simply concatenate.
ii). If adjacent $s_i s_{i+1}$ from the same group G_i, replace them with the result of their product $s_i \circ s_{i+1} \in G_i$ where $\circ$ is the operation for specific G_i.
iii). Remove the product if $s_i \circ s_{i+1} = e_i \in G_i$.

The two definitions are exactly the same but with different mathematical language. All is trying to say the free product group is also a group and it contains all the elements of the groups in our family. It allows us to represent the fundamental group by decomposed the space into many open pieces and remain the non-commutative structure.

Theorem 3.1.3 *(van Kampen) If X is decomposed as the union of path-connected open sets A_α, with each containing the basepoint $x_0 \in X$ and $A_{\alpha_i} \cap A_{\alpha_j}$ is path-connected for any two, then the homomorphism $\phi : *_\alpha \pi_1(A_\alpha) \longrightarrow \pi_1(X)$ is surjective.*

3.2 Categorical van Kampen theorem

One of our purpose is to introduce homotopy theory from categorical language, a potential way to do that is to find the equivalence between different mathematical description to same topological object or theorem. The van Kampen theorem is a good example.

Let $\mathcal{O} = \{U_i\}_{i\in I}$ be a cover of the space X where every U_i is path connected open subsets and $\cap_{i\in K\subseteq I}^{|K|<\infty} U_i \in \mathcal{O}$, which means the intersection of finitely many subsets in $\mathcal{O}$ is again in $\mathcal{O}$. $\mathcal{O}$ can be considered as a category with objects are those open subsets and morphisms are the inclusions between sets. In such way the fundamental groupoid functor restricted to the space and maps in $\mathcal{O}$ sends us to the category of groupoid, that is, $\prod|_{\mathcal{O}} : \mathcal{O} \longrightarrow \mathbf{GP}$. And what van Kampen theorem says in most general categorical words is the fundamental groupoid is the initial objects in the category of all vertex of cones on the groupoid functor.

Theorem 3.2.1 *(General van Kampen Theorem) The groupoid $\prod(X)$ is the colimit of the $\mathcal{O}$-shaped diagram in $\mathbf{GP}$, in mathematical terms:*

$$\prod(X) \cong \mathrm{colim}_{U_i \in \mathcal{O}} \prod(U_i).$$

This work has already be done in [2, Section 2.7], but here we provide a detailed proof and much more clear commutative diagrams.

Proof To construct a proper proof, we have to verify the universal property of colimits. Fundamental groupoid functor restricted on the cover is a map between two categories $\mathscr{O} \longrightarrow \mathbf{GP}$, which is a $\mathscr{O}$-shaped diagram. The theorem is saying $\prod(X)$ is colim$(\prod|_{\mathscr{O}})$. We have to show, $\forall G \in \mathbf{GP}$ and the family of $\phi_i : \prod(U_i) \longrightarrow G$, $\exists!\Phi : \prod(X) \longrightarrow G$ such that $\phi_i = \Phi \circ \iota_i$ for every i, where ι_i is the morphism induced by the inclusions of subsets in category $\mathscr{O}$ under functor $\prod|_{\mathscr{O}}$.

$$\begin{tikzcd}
\prod(U_1) \arrow[rr, "\prod(\hookrightarrow)"] \arrow[dr, "\iota_1"] \arrow[ddr, "\phi_1"'] & & \prod(U_2) \arrow[dl, "\iota_2"'] \arrow[ddl, "\phi_2"] \\
& \operatorname{colim} \prod|_{\mathscr{O}} = \prod(X) \arrow[d, dashed, "\exists!\ \Phi"] & \\
& G &
\end{tikzcd}$$

Our proof includs two steps. First we have to define the morphism $\Phi : \prod(X) \longrightarrow G$ then show it is unique. Be careful that we are working in the category of groupoid, which means any morphism between two objects is a functor, thus our map should both consider the image of objects and morphisms. ob $\prod(X)$ is points of X, $\forall x \in X,\ \exists U_i \in \mathscr{O}\ s.t.\ x \in U_i$. So it is natural to define

$$\Phi_{\text{ob}}(x) := \phi_i(x),\ for\ x \in U_i$$

It is well-defined by the closed of intersection in $\mathscr{O}$. For $x \in U_i \cap U_j$, we have two inclusions $\iota_i : U_i \hookrightarrow U_i \cap U_j$ and $\iota_j : U_i \hookrightarrow U_i \cap U_j$ implies $\phi_i(x) = \phi_{i\cap j}(x) = \phi_j(x)$, so it is independent of the choice of U_i.

The morphism map is somehow similar, notice that mor$(\prod(X))$ are the homotopy class $[f] : x \longrightarrow y$ of the path. As $f([0,1])$ is compact and $\mathscr{O}$ covers X, by Lebesgue's Covering Lemma we have a subdivision $0 = t_0 < t_1 < ... < t_n = 1$ and $\{U_{i1}, U_{i2}, ..., U_{in}\} \subseteq \mathscr{O}$ such that $f([t_{k-1}, t_k]) \subset U_{i_k}$. We get a homotopy subclass on U_{i_k} by restricting our f on the interval $[t_{k-1}, t_k]$ and we get f_k such that $[f] = *_{k=1}^{n}[f_k]$ where f_k is a path in U_{i_k} form x_{k-1} to x_k. After those set up we can define the morphism map of our $\Phi : \prod(X) \longrightarrow G$

$$\Phi_{\text{mor}}[f] := \phi_{i_n}([f_n]) \circ \phi_{i_{n-1}}([f_{n-1}]) \circ ... \circ \phi_{i_2}([f_2]) \circ \phi_{i_1}([f_1])$$

It is independent of the choice of U_i again by the closed intersection of our cover thus it is well-defined. Now we can verify the universal property of the given functor which is the uniqueness of given Φ. Suppose there is another functor $\Psi : \prod(X) \longrightarrow G$ such that $\phi_i = \Psi \circ \iota_i$ for every i.

Consider any $x \in X$, there is $x \in U_i$ for some $U_i \in \mathscr{O}$ which gives the induced inclusion map $\iota_i(x) = x$. So

$$\Psi(x) = \Psi(\iota_i(x)) = \phi_i(x) = \Phi(x)$$

which says the two functor have same image on ob$(\prod(X))$. For any $[f] \in \text{Hom}_{\prod(X)}(x, y)$, we have the same subdivision $[f] = *_{k=1}^{n}[f_k]$, thus $\Psi([f]) = \Psi([f_1] * [f_2] * ... * [f_n]) = \Psi([f_n]) \circ ... \circ \Psi([f_2]) \circ \Psi([f_1])$. Notice that $[f_k]$ in $\text{Hom}_{\prod(U_{i_k})}$ so it is the same after applying the functor induced by the inclusion map, that is $[f_k] = \iota_{i_k}([f_k]) \in \text{Hom}_{\prod(X)}$. In conclusion we have

$$\begin{aligned}
\Psi([f]) &= \Psi([f_n]) \circ ... \circ \Psi([f_2]) \circ \Psi([f_1]) \\
&= \Psi(\iota_{i_n}[f_n]) \circ ... \circ \Psi(\iota_{i_2}[f_2]) \circ \Psi(\iota_{i_1}[f_1]) \\
&= \phi_{i_n}([f_n]) \circ ... \circ \phi_{i_2}([f_2]) \circ \phi_{i_1}([f_1])
\end{aligned}$$

which is exactly $\Phi([f])$. Thus the universal property is verified. $\square$

The detaily proof above is not hard to understand, the key to verify the universal property for categorical objects is to construct a proper one based on what we already have. For those categorical expert one should really easy to follow the proof just by understand four equations of the definitions and verifications. The diagram below review our proof and picture what we are working on so far.

$$\begin{tikzcd}
& x \arrow[rrrr, "{[f]}"] \arrow[dddl, "\Phi"] & & & & y \arrow[dddr, "\Phi"] & \\
\prod(U_x) & & & \prod(X) & & & \prod(U_y) \\
x_1 \arrow[ur, "\iota_x"] \arrow[rr, "{[f_x]}"] \arrow[d, "\phi_x"] & & x_2 \arrow[uul, "\iota_x"'] \arrow[dr, "\phi_x"] & & y_1 \arrow[uur, "\iota_y"] \arrow[rr, "{[f_y]}"] \arrow[dl, "\phi_y"'] & & y_2 \arrow[uul, "\iota_y"'] \arrow[d, "\phi_y"] \\
\Phi(x) = \phi_x(x) \arrow[rrr, "{\phi_x([f_x])}"] \arrow[rrrrrr, bend right, "{\Phi([f])=\Phi([f_x]*[f_y])=\phi_y([f_y])\circ\phi_x([f_x])}"'] & & & \circ \arrow[rrr, "{\phi_y([f_y])}"] & & & \Phi(y) = \phi_y(y) \\
& & & G & & &
\end{tikzcd}$$

Proof by veriying the universal property is not hard but our proof is somehow different because our working is in the category of groupoid **GP**. Every groupoid is a category so the morphism between any two is a functor, which means we have to verify the image both on objects and morphisms in specific groupoid. If we move to the fundamental group version of van Kampen then will be more trivial.

Theorem 3.2.2 *(Categorical van Kampen) The group $\pi_1(X,x)$ is the colimit of the $\mathscr{O}$-shaped diagram restricted on the cover $\pi_1|_{\mathscr{O}} : \mathscr{O} \longrightarrow$ **Grp**, in mathematical words:*

$$\pi_1(X,x) \cong \mathrm{colim}_{U_i \in \mathscr{O}} \pi_1(U_i,x)$$

Proof (Categorical functor proof) The proof follows formally from our previous one but there is some details we need to talk about. Our goal is the same that to verify the universal property, but this time we focus on the category of group **Grp**. Recall we consider group as special groupoid with a single object and the elements of group as the morphisms. We consider the localize functor:

$$\begin{aligned}
&B_{x_0} : \mathbf{GP} \longrightarrow \mathbf{Grp} \\
&\forall G \in \mathrm{ob}(\mathbf{GP}), B_{x_0}(G) = \mathrm{Hom}_G(x_0,x_0) \\
&\forall \mathcal{F} \in \mathrm{Hom}_{\mathbf{GP}}(G,H), B_{x_0}(\mathcal{F}) = \mathcal{F}|_{\mathrm{Hom}_G}(x_0,x_0)
\end{aligned}$$

The definition for this functor on morphism is a restriction of $\mathcal{F}$ on the groupoid G which itself a category, that is, $\mathcal{F}|_{\mathrm{Hom}_G}(x_0,x_0) : \mathrm{Hom}_G(x_0,x_0) \longrightarrow \mathrm{Hom}_H(\mathcal{F}(x_0),\mathcal{F}(x_0))$. By this functor for any elements in our $\mathscr{O}$-shaped diagram, we have $B_{x_0}(\prod(U_i)) = \mathrm{Hom}_{\prod(U_i)}(x_0,x_0) = \pi(U_i,x_0)$, the same to $\prod(X)$. We have a bridge from **GP** and reconsider the last proof, the diagram below commutes

$$\begin{tikzcd}
\mathscr{O} \arrow[r, "\prod"] & \mathbf{GP} \arrow[r, "B_{x_0}"] & \mathbf{Grp}
\end{tikzcd}$$

$$\begin{tikzcd}
U_i \arrow[r, "\prod(*)"] \arrow[d, hook, "\iota"] & \prod(U_i) \arrow[r, "B_{x_0}(*)"] \arrow[d, "\prod(\iota)"] & \pi_1(U_i,x_0) \arrow[d, "B_{x_0}(\prod(\iota))"] \\
U_j \arrow[r, "\prod(*)"] & \prod(U_j) \arrow[r, "B_{x_0}(*)"] & \pi_1(U_j,x_0)
\end{tikzcd}$$

In last proof we show $\prod(X) \cong \mathrm{colim}_{U_i \in \mathscr{O}} \prod(U_i)$ in **GP**, now if our functor B_{x_0} preserves the colimits in **GP** then it is done. In mathematical, we want to show

$$G = B_{x_0}(\mathrm{colim}_{U_i \in \mathscr{O}} \prod(U_i)) \cong \mathrm{colim}_{U_i \in \mathscr{O}} B_{x_0}(\prod(U_i)) = H$$

where $B_{x_0}(\mathrm{colim}_{U_i\in\mathscr{O}}\prod(U_i)) = \pi_1(X,x_0)$ and $\mathrm{colim}_{U_i\in\mathscr{O}}B_{x_0}(\prod(U_i)) = \mathrm{colim}_{U_i\in\mathscr{O}}\pi_1(X,x_0)$, for convenience we use G and H to represent those two groups. Consider the mapping $\theta : H \longrightarrow G$, for any element in H is a loop $[\gamma] \in U_i$ for some $U_i \in \mathscr{O}$. By the inclusion functor J_i from **Grp** to **GP**, $J_i([\gamma]) \in \mathrm{Hom}_{\prod(U_i)}(x_0, x_0)$, then send to □

Proof (Categorical equivalence proof) Now we give another proof totally based on the power of category. For **Gp** and **Grp** there are always two natural functors. One is the inclusion of categories $\mathcal{J} : \pi_1(X,x) \longrightarrow \prod(X)$, which sends a group as one single object $*$ to a specific basepoint x, and sends the homotopy class of loops $[\gamma] \in \mathrm{Hom}_{\pi_1(X,x)}(*,*)$ to the automorphism of x in $\prod(X)$. As a dual there is a contract functor $\mathcal{F} : \prod(X) \longrightarrow \pi_1(X,x)$ with a set up that $\forall y \in X$ there exists a chosen path $\alpha_y : x \longrightarrow y$ such that $\alpha_x = c_x$ is constant path, *i.e.* $f(t) = x$, $\forall t \in [0,1]$. Moreover, for $y \in U_i$ for some $U_i \in \mathscr{O}$ lies entirely in U_i. These two functors are called categorical equivalence functors and each of them is the inverse equivalence to the other, which one should easy to verify $\mathcal{F} \circ \mathcal{J} = \mathrm{Id}_{\mathbf{Grp}}$.

We shall assume our cover $\mathscr{O}$ is finite and then general to infinite case. And the idea of proof is simple as the commutative diagram shows below, which we use $\mathcal{F}$ and $\mathcal{J}$ to build up a bridge between two colimts.

$$\begin{tikzcd}
 & \pi_1(U_1,x_0) \arrow[dddd, hook, "\pi_1(\prod(\hookrightarrow))"'] \arrow[ddr, "i_1"] \arrow[ddrrr, "\psi_1"] & & & \\
\prod(U_1) \arrow[ur, "\mathcal{F}_{U_1}"] \arrow[dd, hook, "\prod(\hookrightarrow)"'] \arrow[drrr, hook, "\iota_1"] \arrow[drrrr, "\phi_1"] & & & & \\
 & & \pi_1(X,x_0) \arrow[r, "\mathcal{J}"] & \prod(X) \arrow[r, "\exists!\Phi"] & G \\
\prod(U_2) \arrow[dr, "\mathcal{F}_{U_2}"'] \arrow[urrr, hook, "\iota_2"] \arrow[urrrr, "\phi_2"] & & & & \\
 & \pi_1(U_2,x_0) \arrow[uur, "i_2"] \arrow[uurrr, "\psi_2"'] & & &
\end{tikzcd}$$

For any $y \in U_i$ by our set up to pick path $x \longrightarrow y$ entirely in U_i, thus the functors travels three categories

$$\prod(U_i) \xrightarrow{F_{U_i}} \pi_1(U_i,x_0) \xrightarrow{\psi_i} G$$

is an $\mathscr{O}$-shaped diagram $\psi_i \circ F_{U_i} : \prod|\mathscr{O} \longrightarrow \mathbf{GP}$, notice that a group could also be considered as groupoid. By the groupoid version of van Kampen, there exists a unique map in category of **GP** that $\Phi : \prod(X) \longrightarrow G$ such that $\psi_i \circ \mathcal{F}_{U_i} = \Phi \circ \iota_i$ for all i. By the uniqueness of Φ and the bridge $\mathcal{J} : \pi(X,x_0) \longrightarrow \prod(X)$, we have a unique homomorphism $\Psi = \Phi \circ \mathcal{J}$ as required, which satisfies the universal property of colimits that $\Psi \circ i_i = \psi_i$, where i_i is the inclusion from $\pi_1(U_i,x_0)$ to $\pi(X,x_0)$ induced be the inclusion as sets, and ψ_i is the map $\pi_1(U_i,x_0)$ to G similar to ϕ_i before in **GP**.

Our work in not done yet, we have to general to infinite case. For a infinite cover $\mathscr{O}$ we have $\mathscr{F}$ as the set of those finite subsets in $\mathscr{O}$ which closed under finite intersection. For specific subset $S \in \mathscr{F}$, we denote the union of $U_i \in S$ as U_S and it is clear that S is a cover of U_S. Moreover the $\mathscr{F}$ is again a finite cover can be considered as a category of $\mathscr{O}$ in finite case, so we have

$$\mathrm{colim}_{U_i\in S}\pi_1(U_i,x_0) \cong \pi_1(U_S,x_0)$$
$$\mathrm{colim}_{S_i\in\mathscr{F}}\pi_1(U_{S_i},x_0) \cong \pi_1(X,x_0)$$

Now for any infinite cover $\mathscr{O}$ we subdivide to $\mathscr{F}$ and by arguement before any loops in X has image in some U_S, so all we need to prove is

$$\text{colim}_{U_i\in\mathscr{O}}\pi_1(U_i,x_0)\cong\text{colim}_{S_i\in\mathscr{F}}\pi_1(U_{S_i},x_0)\cong\text{colim}_{S\in\mathscr{F}}\text{colim}_{U_i\in S}\pi_1(U_i,x_0)$$

The iterated colimt is isomorphic to the single colimit $\text{colim}_{(U_i,S_i)\in(\mathscr{O},\mathscr{F})}\pi_1(U_i,x_0)$. The category $(\mathscr{O},\mathscr{F})$ has morphism $(U_1,S_1)\longrightarrow(U_2,S_2)$ if both $U_1\subset U_2$ and $U_{S_1}\subset U_{S_2}$. So there is a natural inclusion functor that $\mathcal{I}:\mathscr{O}\longrightarrow(\mathscr{O},\mathscr{F})$ where the diagram below commutes

$$\begin{tikzcd}
 & U_1 \arrow[r, "\mathcal{I}"] \arrow[dd, hook] & (U_1,\{U_1\}) \arrow[dd, hook] & \\
\mathscr{O} & & & (\mathscr{O},\mathscr{F}) \\
 & U_2 \arrow[r, "\mathcal{I}"] & (U_2,\{U_2\}) &
\end{tikzcd}$$

where $\mathcal{I}$ sends U_i to its singleton set $\{U_i\}$ in $\mathscr{F}$. One should easily verify the only difference between $\mathscr{O}$ and $(\mathscr{O},\mathscr{F})$ is that for a homomorphisms $\pi_1(U_1,x_0)\longrightarrow\pi_1(U_2,x_0)$, it only applys in $\mathscr{O}$ once but many times in $(\mathscr{O},\mathscr{F})$ with the same result comes out, there is no new contribution to our colimit. On the other side is more trivial, by projection gives us the function $\mathcal{P}:(\mathscr{O},\mathscr{F})\longrightarrow\mathscr{O}$. Those functors composite with $\pi_1:\mathscr{O}\longrightarrow\mathbf{Grp}$ gives us the isomorphism

$$\text{colim}_{U_i\in\mathscr{O}}\pi_1(U_i,x_0)\cong\text{colim}_{(U_i,S_i)\in(\mathscr{O},\mathscr{F})}\pi_1(U_i,x_0)$$

$$\begin{tikzcd}
 & \pi_1(U_1,x_0) \arrow[ddl, "\iota_1"'] \arrow[dddd, hook] & & & \pi_1(U_1,\{U_1\}) \arrow[ddr, "(\iota_1,I_1)"] \arrow[dddd, hook] & \\
 & & U_1 \arrow[ul, "\pi_1"'] \arrow[r, shift left, "\mathcal{I}"] \arrow[dd, hook] & (U_1,\{U_1\}) \arrow[l, shift left, "\mathcal{P}"] \arrow[ur, "\pi_1"] \arrow[dd, hook] & & \\
\text{colim}_{U_i\in\mathscr{O}}\pi_1(U_i,x_0) \arrow[rrrrr, shift left] & & & & & \text{colim}_{(U_i,S_i)\in(\mathscr{O},\mathscr{F})}\pi_1(U_i,x_0) \arrow[lllll, shift left] \\
 & & U_2 \arrow[dl, "\pi_1"] \arrow[r, shift left, "\mathcal{I}"] & (U_2,\{U_2\}) \arrow[l, shift left, "\mathcal{P}"] \arrow[dr, "\pi_1"'] & & \\
 & \pi_1(U_2,x_0) \arrow[uul, "\iota_2"] & & & \pi_1(U_2,\{U_2\}) \arrow[uur, "(\iota_2,I_2)"'] &
\end{tikzcd}$$

The commutative diagram shows clearly how two easy functors between $\mathscr{O}$ and $(\mathscr{O},\mathscr{F})$ build up the isomorphism. And our prove end up with

$$\text{colim}_{U_i\in\mathscr{O}}\pi_1(U_i,x_0)\cong\text{colim}_{(U_i,S_i)\in(\mathscr{O},\mathscr{F})}\pi_1(U_i,x_0)\cong\text{colim}_{S_i\in\mathscr{F}}\pi_1(U_{S_i},x_0)\cong\pi_1(X,x_0) \qquad \square$$

What's so intersting is that to be reversed, recall that $\mathcal{F}:\prod(X)\longrightarrow\pi_1(X,x_0)$. If we have uniquely $\Psi:\pi_1(X,x_0)\longrightarrow G$ restricts to ψ_i on each $\pi_1(U_i,x_0)$, then $\Psi\circ\mathcal{F}:\prod(X)\longrightarrow G$ restricts to $\psi_i\circ\mathcal{F}_{U_i}$ which is exactly ϕ_i, meaning it is the way to prove groupoid version if we can prove group version constructivly, the van Kampen in groupoid and groups are equivalent.

Chapter 4

Galois connections

Now we will introduce a fantastic concept which is likely a univseral property in many correspondence relation, which is the Galois connection. The categorical equivalence of Galois theory and covering space theory can be hold is actually that they are the special case of Galois connection in algebra and topology.
While we begin with the elementary definitions of Galois connections as found in Wikipedia [5], our main contribution lies in providing rigorous examples and extending these concepts to more general settings.

4.1 Monotone and Antitone Galois Connection

A Galois connection is a particular correspondence between two partially ordered sets, or posets which as defined below.

Definition 4.1.1 (Posets) A partial ordered set or poset is a pair $(P, \leq)$ consists a set P and a binary operation leq called partial relation and satisfying following axioms:
(i) Reflexivity: $\forall a \in P, a \leq a$. (ii) Antisymmetry: $\forall a, b \in P$, $a \leq b$ and $\leq a$ implies $a = b$. (iii) Transitivity: $\forall a, b, c \in P$, $a \leq b$ and $b \leq c$ implies $a \leq c$.

We say two element a and b in the poset P are comparable if $a \leq b$ or $b \leq a$ otherwise they are incomparable. If all elements are comparable then the parital ordered set is a totally ordered set.

Example 4.1.2 Easy to verify $(\mathbb{R}, \leq)$ is a totally ordered set.

Example 4.1.3 Take $P = \mathcal{P}(\{1, 2, 3\})$ as example, $(P, \subseteq)$ is a poset not totally ordered because $\{1\}$ and $\{2\}$ are incomparable.

Example 4.1.4 $(\mathbb{N}^+, |)$ is a parital ordered set but it is not totally ordered since $2 \nmid 3$.

Definition 4.1.5 (preordered sets) A preordered set $(P, \leq)$ is a poset whithout the antisymmetry.

Example 4.1.6 $(N, |)$ is a preordered but not partial ordered, consider 2 and -2.

Definition 4.1.7 (Monotone Galois connection)[5] A monotone Galois connection consists two monotone functions between two posets $(A, \leq)$ and $(B, \leq)$ where $F : A \longrightarrow B$ and $G : B \longrightarrow A$ such that

$$\forall a \in A, b \in B, we\ have\ F(a) \leq b \Longleftrightarrow a \leq G(b)$$

We call F the lower adjoint of G and G the upper adjoint of F. In posets we say there is a morphism $a \longrightarrow b$ if and only if $a \leq b$, so in diagram we have

$$
\begin{tikzcd}
\bullet \arrow[r, dashed] & \bullet \arrow[r, dashed] & a \arrow[r, "\leq"] \arrow[d, "F"] & G(b) \arrow[r, dashed] & \bullet \arrow[r, dashed] & \bullet \\
\bullet \arrow[r, dashed] & \bullet \arrow[r, dashed] & F(a) \arrow[r, "\leq"] & b \arrow[r, dashed] \arrow[u, "G"] & \bullet \arrow[r, dashed] & \bullet
\end{tikzcd}
$$

You should easily have the intuition that F and G are not inverse of each other otherwise the Galois connection will be trivial. Or in other words an upper or lower adjoint of a Galois connection uniquely determines the other. Thus we have another definition of Galois connection, or, a proposition followed immediately by the original definition.

Proposition 4.1.8 *Let $(F : A \longrightarrow B, G : B \longrightarrow A)$ be the Galois connection between posets $(A, \leq_A)$ and $(B, \leq_B)$. Then $\forall a \in A, b \in B$, we have*

$$F(a) = \min\{b \in B | a \leq_A G(b)\},$$
$$G(b) = \max\{a \in A | F(a) \leq_B b\}.$$

Proof We only need to prove one side of the Galois connection and the other side comes dually. To show $F(a) = \min\{b \in B|a \leq_A G(b)\}$, is to show $F(a)$ is the minimal element of the set $S_a = \{b \in B|a \leq_A G(b)\}$. By the definition of Galois connection we take $b = F(a)$ then

$$F(a) \leq_B F(a) \Longleftrightarrow a \leq_A G(F(a))$$

where left side always holds so does the right hand side, so $a \leq_A G(F(a))$, which is $F(a) \in S_a$. Now we want to show $F(a)$ is greatest lower bound of this set
Take any $b' \in S_a$, by the definition of S_a we have $a \leq_A G(b')$ implies $F(a) \leq_B b$, thus $F(a)$ is the lower bound. Moreover, let $b_0 \in S_a$ be any lower bound that $\forall b \in S_a, b_0 \leq_B b$, we take $b = F(a) \in S_a$ then $b_0 \leq_B F(a)$, with $F(a) \leq_B b_0$ we have $b_0 = F(a)$ by antisymmetry. And the other side is dual.□

A consequence of this is that if F or G is bijective then $F = G^{-1}$ and the Galois connection is just an isomorphism of posets. So the Galois connection is a weaker version of isomorphism. And you may notice that actually the composition of two functions are the automorphism of the posets.

Definition 4.1.9 (Closure operator and kernel operator) A closure operator is the composition of the Galois connection by first applying the lower adjoint as $GF : A \longrightarrow A$ and the dual is called the kernel operator $FG : B \longrightarrow B$ which first applying the upper adjoint.

Proposition 4.1.10 *The closure operator GF and kernel operator FG are both monotone, idempotent and extensive (or contractive).*

Proof We say a operator C on a poset is monotone if $\forall a, b \in P, a \leq b$ implies $C(a) \leq C(b)$, and the two are monotone natrually by they are composition of monotone functions. We say a operator C is idempotent if $C^2 = Id$. It is easy to verify $GF(GF(a)) = a, \forall a \in A$ and so does the other one. We say GF is extensive if $\forall a \in A, a \leq GF(a)$ and FG is contractive if $\forall b \in B, FG(b) \leq b$, which will be proved in the next proposition. □

Proposition 4.1.11 *$\forall a \in A, b \in B$, we have $a \leq_A GF(a)$ and $FG(b) \leq_B b$.*

Proof Take $b = F(a)$ in the Galois connection condition $F(a) \leq_B b \Longleftrightarrow a \leq_A G(b)$. Since $F(a) \leq_B F(a)$ holds trivially, we obtain $a \leq_A G(F(a))$. The other side is dual. □

From the extensive and contractive property we can see we may lose information after applying the Galois connection, which is another reason why Galois connection is weaker than isomorphism. Thus we define a special Galois connection called Galois insertion or Galois embedding.

Definition 4.1.12 (Galois embedding) A Galois connection (F, G) is a Galois embedding if the kernel operator is the identity, in mathematical

$$FG(b) = b, \forall b \in B$$

Proposition 4.1.13 *The following conditions are equivalent for a Galois connection (F, G) between posets $(A, \leq_A)$ and $(B, \leq_B)$: (i) (F, G) is a Galois embedding. (ii) G is injective. (iii) F is surjective.*

Proof $(i) \Rightarrow (ii)$: Supp $G(b_1) = G(b_2)$, then $FG(b_1) = FG(b_2)$ implies $b_1 = b_2$ by Galois embedding, thus G is an injection.
$(ii) \Rightarrow (iii)$: For any $b \in B$ we have $FG(b) \leq_B b$ which $G(FG(b)) \leq_A G(b)$ by monotonicity of G. And we know $G(b) \leq GF(G(b))$ because GF is extensive, so $GF(G(B)) = G(b)$ by antisymmetry. By the injectivity of G we have $F(G(b)) = b$, thus $\forall b \in B$ take $a = G(b) \in A$ we have $F(a) = b$, F is surjective.
$(iii) \Rightarrow (i)$: By surjectivity for any $b \in B$ we pick $a \in A$ such that $F(a) = b$. Then $F(a) = b \leq_B b$ always holds, implies $a \leq_A G(b)$. Thus $b = F(a) \leq_B F(G(b))$ by monotonicity of F, and we know $FG(b) \leq_B b$ because FG is contractive, so $FG(b) = b$ by antisymmetry. $\square$

Now we give planty of examples of Galois connection in different mathematical fields.

Example 4.1.14 (Bijection) Bijection between two sets can be considered as the trivial Galois connection where both lower and upper adjoint are the bijective function. $(X, =)$ and $(Y, =)$ are two posets with equality relation that $f(x) = y \iff x = f^{-1}(y) = g(y)$. Also, it is a Galois embedding.

Example 4.1.15 (Floor function) Consider $A = (\mathbb{Z}, \leq)$ and $B = (\mathbb{R}, \leq)$, we have the inclusion $F : \mathbb{Z} \hookrightarrow \mathbb{R}$ and $G : \mathbb{R} \longrightarrow \mathbb{Z}$ defined by the floor function $G(x) = \lfloor x \rfloor$. It is easy to verify this is a Galois connection since $F(n) = n \leq x \iff n \leq \lfloor x \rfloor = G(x)$. However, it is not a Galois embedding since $FG(x) = \lfloor x \rfloor \neq x$ in general.

Example 4.1.16 (Ceiling function) Moreover, the dual given by the ceiling function is also a Galois connection as $G(x) = \ulcorner x \urcorner$ then we have $G(x) \leq n \iff x \leq F(n)$ implies (G, F) is a Galois connection between (B, A), and this is a Galois embedding as $GF(n) = n, \forall n \in \mathbb{Z}$. (Notice that to satisfy Galois connection condition we have to reverse the postes and the order of two functions, which makes it a Galois embedding).

Example 4.1.17 (Prime factors function) Let $A = (\mathbb{N}, \leq)$ and $B = (\mathbb{N}, |)$. For a fixed prime number p we difine two monotone function $F : A \longrightarrow B$ and $G : B \longrightarrow A$ as

$$F(n) = p^n, G(m) = v_p(m)$$

where $v_p(m)$ is the largest exponent k such that p^k divides m. It is easy to verify this is a Galois connection since $F(n) = p^n | m \iff n \leq v_p(m) = G(m)$. However, it is not a Galois embedding since $FG(m) = p^{v_p(m)} \neq m$ in general.

Example 4.1.18 (Ideal containment and radical operation) Consider a commutative ring R. with unity, let A be the poset of all ideals and B be the posets of radical ideals both ordered by inclusion $\subseteq$. We have the maps $F : A \longrightarrow B$ and $G : A \hookrightarrow B$ a Galois connection as

$$F(I) = \sqrt{I} = \{r \in R |\ r^n \in I\ for\ some\ n\ \geq 1\}$$

and $G(J) = J$ the inclusion. They satisfy the Galois connection because $F(I) = \sqrt{I} \subseteq J \iff I \subseteq J = G(J)$. Moreover, it is a Galois embedding since $GF(J) = \sqrt{J} = J$ for any radical ideal J.

Example 4.1.19 (Supremum and Upper Level Sets) Let X be any non-empty set and $f : X \longrightarrow \mathbb{R}$ be any function. Consider $A = (\mathcal{P}(X), \subseteq)$ and $B = (\mathbb{R} \cup \{\infty\}, \leq)$. We have $F : A \longrightarrow B$ and $G : B \longrightarrow A$ a Galois connection as

$$F(S) = \sup_{x \in S} f(x)$$
$$G(t) = \{x \in X | f(x) \leq t\}$$

because $F(S) = \sup_{x \in S} f(x) \leq t \Longleftrightarrow S \subseteq \{x \in X | f(x) \leq t\} = G(t)$. However, it is not a Galois embedding since $FG(t) = \sup_{x \in \{x \in X | f(x) \leq t\}} f(x) \neq t$ in general.

Example 4.1.20 (Partitions and Equivalence Realations) Let X be any non-empty set. We have

$$A = (\mathrm{Part}(X), \leq_{\mathrm{ref}})$$
$$B = (\mathrm{EqRel(X)}, \subseteq)$$

which A is the set of all partitions of X ordered by refinement: $r_1 \leq r_2$ if every block of r_1 is contained in some block of r_2 (i.e., $\{\{1,2\},\{3\}\} \leq_{\mathrm{ref}} \{\{1,2,3\}\}$ but $\{\{1,2\},\{3\}\} \nleq_{\mathrm{ref}} \{\{1\},\{2,3\}\}$). And B is the set of all equivalence realations on X ordered by inclusion. You may studied this before when you learned abstract algebra that there is a bijection between partitions and equivalence relations. Now we define two monotone functions $F : A \longrightarrow B$ and $G : B \longrightarrow A$ as

$$F(r) = R_r = \{(x,y)| \ x \ and \ y \ belong \ to \ the \ same \ block \ of \ r\}$$
$$G(R) = X/R = \{[x]_R | \ x \in X\}$$

Then (F, G) is a Galois connection that $F(r) = R_r \subseteq R \Longleftrightarrow r \leq_{\mathrm{ref}} X/R = G(R)$. And it is a Galois embedding since $FG(R) = R_{X/R} = R$ for any equivalence relation R, moreover $GF(r) = X/R_r = r$ for any partition r so we have an isomorphism between these two posets.

Now you may understand the power of Galois connection which occurs everywhere in different mathematical fields, and it reveals the deep correspondence between two posets. But in general, the corresponding between two posets are not always in same direction, that is to say, the monotone condition may not hold. Thus we have to generalize our Galois connection to antitone case.

4.2 Antitone Galois connection

Many important examples of Galois connections are antitone, what is interesting is that the motivation of Galois connection at very beginning is actually goes antitone, which mathematician Evariste Galois found the correspondence between the subgroups of Galois group and the subextensions of field extension is an antitone connection, and this reversing-correspondence pattern occurs everywhere in mathematics so we generalize Galois theory to Galois connection.

Definition 4.2.1 (Antitone Galois connection)[5] An antitone Galois connection considered two antitone functions between two posets $(A, \leq)$ and $(B, \leq)$ where $F : A \longrightarrow B$ and $G : B \longrightarrow A$ such that

$$\forall a \in A, b \in B, we \ have \ b \leq F(a) \Longleftrightarrow a \leq G(b)$$

We call F the upper polarity of G and G the lower polarity of F. Which is the dual to **Definition 4.1.7** and in diagram we have

$$\begin{tikzcd}
\bullet \arrow[r, dashed] & \bullet \arrow[r, dashed] & a \arrow[r, "\leq"] \arrow[d, "F"] & G(b) \arrow[r, dashed] & \bullet \arrow[r, dashed] & \bullet \\
\bullet & \bullet \arrow[l, dashed] & F(a) \arrow[l, dashed] & b \arrow[l, "\leq"] \arrow[u, "G"'] & \bullet \arrow[l, dashed] & \bullet \arrow[l, dashed]
\end{tikzcd}$$

with the morphism from $F(a)$ to b in B reversed to b to $F(a)$.

Everything we have done in monotone Galois connection can be dualized to antitone case.

Proposition 4.2.2 *Let $(F : A \longrightarrow B, G : B \longrightarrow A)$ be the antitone Galois connection between posets $(A, \leq_A)$ and $(B, \leq_B)$. Then $\forall a \in A, b \in B$, we have*

$$F(a) = \max\{b \in B | a \leq_A G(b)\},$$
$$G(b) = \max\{a \in A | b \leq_B F(a)\}.$$

Proof Dualize the proof of **Proposition 3.1**. □

Proposition 4.2.3 *$\forall a \in A, b \in B$, we have $a \leq_A GF(a)$ and $b \leq_B FG(B)$.*

Proof Dualize the proof of **Proposition 3.3**. □

In antitone Galois connection, the operator $GF : A \longrightarrow A$ and $FG : B \longrightarrow B$ are both monotone, idempotent and extensive (or contractive) as well. And they are both called closure operator since any element will be mapped to a smaller one after applying the operator.
Now we are able to study some examples of antitone Galois connection, which are more interesting than monotone case.

Example 4.2.4 (Inverse function) The simplest example as $A = B = (\mathbb{R}, \leq)$ with $F(x) = -x$ and $G(y) = -y$. They have satisfy the antitone Galois connection condition since $-x \leq y \iff x \leq -y$. Moreover, it is a Galois embedding and an isomorphism since $FG(y) = y$ and $GF(x) = x$.

Example 4.2.5 (Orthogonal complement) Consider a finite dimensional inner product space V over $\mathbb{R}$, let $A = B = (\mathrm{Sub}(V), \subseteq)$ be the poset of all subspaces of V ordered by inclusion. We have $F : A \longrightarrow B$ and $G : B \longrightarrow A$ defined by the orthogonal complement operation

$$F(U) = U^{\perp} = \{v \in V|\ \langle v, u\rangle = 0, \forall u \in U\}$$
$$G(W) = W^{\perp} = \{v \in V|\ \langle v, w\rangle = 0, \forall w \in W\}$$

They satisfy the antitone Galois connection because $W \subseteq U^{\perp} \iff U \subseteq W^{\perp}$. Moreover, it is a Galois embedding since $FG(W) = W^{\perp\perp} = W$ for any subspace W, and $GF(U) = U^{\perp\perp} = U$ for any subspace U, thus we have an isomorphism between these two posets.

After studying these elementary examples, now we will introduce some advanced applications of antitone Galois connection in different mathematical fields or even logical fields, to demonstrate the power of it.

Example 4.2.6 (Hilbert's Nullstellensatz) We introduce a famous theorem that plays the role of cornerstone in algebraic geometry as an antitone Galois connection. Consider classical algebraic geometry over an algebraically closed field k, which means every non-constant polynomial in $k[x]$ has a root in k; in mathematical terms:

$$\forall f \in k[x] \setminus k,\ \deg f \geq 1 \Longrightarrow \exists \alpha \in k \text{ s.t. } f(\alpha) = 0.$$

Based on the field k we have two posets

$$A = (\mathcal{P}(k[x_1, x_2, \ldots, x_n]), \subseteq),$$
$$B = (\mathcal{P}(k^n), \subseteq)$$

where $k[x_1, x_2, \ldots, x_n]$ is the polynomial ring in n variables over k, and k^n is the n-dimensional affine space over k.

We define the vanishing mapping $F : A \longrightarrow B$ and the ideal mapping $G : B \longrightarrow A$ as

$$F(S) = V(S) = \{P = (a_1, a_2, \ldots, a_n) \in k^n \mid \forall f \in S,\ f(a_1, a_2, \ldots, a_n) = 0\},$$
$$G(V) = I(V) = \{f \in k[x_1, x_2, \ldots, x_n] \mid \forall P = (a_1, a_2, \ldots, a_n) \in V,\ f(a_1, a_2, \ldots, a_n) = 0\}.$$

The pair (F, G) forms an antitone Galois connection since $V \subseteq V(S) \iff S \subseteq I(V)$. The famous Hilbert's Nullstellensatz (in its strong form) states that for any ideal I in $k[x_1, x_2, \ldots, x_n]$, we have $I(V(I)) = \sqrt{I}$, where $\sqrt{I}$ is the radical of the ideal I. Thus the closure operator $GF : A \longrightarrow A$ is exactly the radical operation on ideals: $GF(I) = I(V(I)) = \sqrt{I}$. For any algebraic subset $V \subseteq k^n$ (i.e., $V = V(J)$ for some ideal J), we have $V(I(V)) = V$, which means that on the sub-poset of algebraic sets, the kernel operator $FG : B \longrightarrow B$ acts as the identity.
Therefore, when restricted to radical ideals and algebraic varieties, (F, G) induces a Galois embedding, in fact, an order-isomorphism. This Galois connection builds the bridge between algebraic geometry and commutative algebra, and reveals the deep correspondence between algebraic sets in affine space and radical ideals in the polynomial ring.

Example 4.2.7 (Closure-interior duality in topology) In point-set topology, the operations of taking the closure and the interior of a subset of a topological space are fundamental dual notions. This duality can be expressed elegantly as an antitone Galois connection. We have (X, τ) which a topological space. Consider two partially ordered sets with same elements but different inclusion direction that

$$A = (\mathcal{P}(X), \subseteq),$$
$$B = (\mathcal{P}(X), \supseteq).$$

We easily have $T_1 \leq_B T_2$ if and only if $T_1 \supseteq T_2$. Thus B is the order-dual of A. Now we define two maps $F : A \longrightarrow B$ and $G : B \longrightarrow A$ by

$$F(S) = X \setminus S^\circ,$$
$$G(T) = X \setminus \overline{T},$$

where S° denotes the interior of S and $\overline{T}$ the closure of T. The pair (F, G) is an antitone Galois connection: for any $S \in A$ and $T \in B$, $T \supseteq F(S) \iff S \subseteq G(T)$. In terms of topological operations, this is equivalent to

$$\overline{S} \subseteq T \iff S \subseteq (X \setminus T)^\circ,$$

a basic lemma in topology stating that the closure of S is contained in T exactly when S lies inside the interior of the complement of T. Moreover, the composite operators have clear topological meanings:

$$GF(S) = G(F(S)) = X \setminus \overline{X \setminus S^\circ} = S^\circ,$$
$$FG(T) = F(G(T)) = X \setminus (X \setminus \overline{T})^\circ = \overline{T}.$$

Thus $GF : A \to A$ is the interior operator, and $FG : B \to B$ is the closure operator. As a concrete illustration, take $X = \mathbb{R}$ with the usual topology, $S = (0, 1)$, and $T = [0.5, 1.5]$. Then $F(S) = \mathbb{R} \setminus (0, 1)^\circ = \mathbb{R} \setminus (0, 1) = (-\infty, 0] \cup [1, \infty), G(T) = \mathbb{R} \setminus \overline{[0.5, 1.5]} = \mathbb{R} \setminus [0.5, 1.5] = (-\infty, 0.5) \cup (1.5, \infty)$. One checks that $T \supseteq F(S)$ is false (since $0.7 \in T$ but $0.7 \notin F(S)$) and $S \subseteq G(T)$ is false as well, confirming the equivalence. This Galois connection captures the perfect duality between closure and interior in topology. It shows that enlarging a set (in the sense of taking its closure) corresponds, under complementation, to shrinking its complement (by taking the interior), and vice versa—a precise order-theoretic formulation of a familiar topological phenomenon.

Now we introduce some logical examples which more abstract but very interesting.

Example 4.2.8 (Formal concept analysis) A formal context is a triple (G, M, I) where G is the set of objects, M is the set of attributes, and $I \subseteq G \times M$ is a binary relation indicating which objects have which attributes. For instance, $(g, m) \in I$ means that objects g has attribute m. Now we construct two posets as

$$A = (\mathcal{P}(G), \subseteq),$$
$$B = (\mathcal{P}(M), \subseteq)$$

And we have a mapping called intersion $F : A \longrightarrow B$ and a extension $G : B \longrightarrow A$ defined on any $S \subseteq G$ ang $T \subseteq M$ as

$$F(S) = S' = \{m \in M |\ \forall g \in S, (g, m) \in I\}$$
$$G(T) = T' = \{g \in G |\ \forall m \in T, (g, m) \in I\}$$

Which means F gives all common attributes of objects in S, for example $F(\{apple, banana\}) = \{fruit,$
$edible, ...\}$; and G gives all objects that have all attributes in T, for example $G(\{fruit, edible\}) = \{apple, banana, orange, ...\}$. Then (F, G) is an antitone Galois connection since $T \subseteq S' \iff S \subseteq T'$. Moreover, the closure operators $GF : A \longrightarrow A$ and $FG : B \longrightarrow B$ are both closure operators in the sense of formal concept analysis. The closed sets under these operators are called formal concepts, which form a complete lattice known as the concept lattice of the formal context. This Galois connection provides a powerful framework for data analysis and knowledge representation.

Example 4.2.9 (Propositional logic of syntax and semantics) Propositional logic can be introduced as a formal system with a set of well-formed formulas (WFFs) constructed from propositional variables and logical connectives. We have a deep duality between syntax and semantics, where syntax refers to the formal structure of formulas, and semantics refers to their meaning or truth values in interpretations. This duality is captured precisely by an antitone Galois connection.

Let λ be a propositional language with propositional variables $p_1, p_2, p_3, \dots$ and logical connectives $\neg, \wedge, \vee, \rightarrow$. For example, $p_1 \wedge (p_2 \vee \neg p_3)$ is a well-formed formula in λ. We define two posets:

$$A = (\mathcal{P}(\mathrm{Form}(\lambda)), \subseteq),$$
$$B = (\mathcal{P}(\mathrm{Mod}(\lambda)), \subseteq),$$

where A consists of all sets Γ of well-formed formulas in λ, ordered by inclusion, and B consists of all sets $\mathcal{M}$ of models (interpretations) of λ, ordered by inclusion. Now we can define two mappings $F : A \longrightarrow B$ and $G : B \longrightarrow A$ by

$$F(\Gamma) = \mathrm{Mod}(\Gamma) = \{\mathcal{M} \in \mathrm{Mod}(\lambda) \mid \mathcal{M} \models \phi \text{ for all } \phi \in \Gamma\},$$
$$G(\mathcal{M}) = \mathrm{Th}(\mathcal{M}) = \{\phi \in \mathrm{Form}(\lambda) \mid \mathcal{M} \models \phi \text{ for all } \mathcal{M} \in \mathcal{M}\}.$$

Here $\mathcal{M} \models \phi$ means the model $\mathcal{M}$ satisfies the formula ϕ. The pair (F, G) forms an antitone Galois connection that for every $\Gamma \in A$ and $\mathcal{M} \in B$ we have $\mathcal{M} \subseteq F(\Gamma) \iff \Gamma \subseteq G(\mathcal{M})$. Indeed, if $\mathcal{M} \subseteq \mathrm{Mod}(\Gamma)$, then each $\phi \in \Gamma$ holds in every model of $\mathcal{M}$, hence $\phi \in \mathrm{Th}(\mathcal{M})$; the converse follows similarly. Moreover, two closure operators arise naturally. On the syntactic side we got $GF(\Gamma) = \mathrm{Th}(\mathrm{Mod}(\Gamma)) = \{\phi \mid \Gamma \models \phi\}$ is the set of all logical consequences of Γ. On the semantic side, $FG(\mathcal{M}) = \mathrm{Mod}(\mathrm{Th}(\mathcal{M}))$ consists of all models that satisfy exactly the same formulas as those in $\mathcal{M}$.

Now we introduce a concrete instance with only two propositional variables p and q. There are exactly four possible models:

$$\begin{aligned}\omega_1 &: p = \mathrm{T}, q = \mathrm{T},\\ \omega_2 &: p = \mathrm{T}, q = \mathrm{F},\\ \omega_3 &: p = \mathrm{F}, q = \mathrm{T},\\ \omega_4 &: p = \mathrm{F}, q = \mathrm{F}.\end{aligned}$$

Take $\Gamma = \{p\}$. Then $F(\Gamma) = \mathrm{Mod}(p) = \{\omega_1, \omega_2\}$, because only ω_1 and ω_2 satisfy p. For the model set $\mathcal{M} = \{\omega_1, \omega_2\}$ we have

$$G(\mathcal{M}) = \mathrm{Th}(\{\omega_1, \omega_2\}) \supseteq \{p, p \vee q, p \vee \neg q, \neg\neg p, \ldots\}.$$

One verifies directly that $\mathcal{M} \subseteq F(\Gamma)$ and $\Gamma \subseteq G(\mathcal{M})$, confirming the Galois condition. Moreover, $GF(\Gamma) = \mathrm{Th}(\{\omega_1, \omega_2\}) = \{\phi \mid p \models \phi\}$, the set of all consequences of p, and $FG(\mathcal{M}) = \mathrm{Mod}(\mathrm{Th}(\{\omega_1, \omega_2\})) = \{\omega_1, \omega_2\}$, showing that $\{\omega_1, \omega_2\}$ is already closed.
This example formalizes the essential duality between syntax (formulas and deduction) and semantics (models and truth). The connection shows that enlarging a set of formulas restricts the class of models that satisfy them, while enlarging a class of models weakens the theory that holds in all of them.

It is ok that if you can not understand all examples we have introduced here. All the works of examples is trying to show how powerful the Galois connection is especially the antitone version. And what we will do next if obvious as Galois connection appears everywhere in different mathematical fields or even logic.

When we formalize the definition of Galois connection, one fact will shocks you is that this is actually the definition of adjoint functors in category theory! Or, maybe you will not be impressed if you got superior mathematical intuition and have the feeling of it already.

4.3 Categorical Galois Connections

We now generalize our Galois connection to have better view in different mathematical fields, to formalize it, we have

Definition 4.3.1 (Categorical monotone Galois connection) For categories $\mathscr{C}$ and $\mathscr{D}$ with the functors $\mathcal{F} : \mathscr{C} \longrightarrow \mathscr{D}$ and $\mathcal{G} : \mathscr{D} \longrightarrow \mathscr{C}$. We say $\mathcal{F}$ and $\mathcal{G}$ satisfy th Galois connection if there exists a natural transformation

$$\Phi : \mathrm{Hom}_{\mathscr{D}}(\mathcal{F}(-), -) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(-, \mathcal{G}(-))$$

which $\forall c \in \mathscr{C}, d \in \mathscr{D}$, gives a bijection

$$\Phi_{c,d} : \mathrm{Hom}_{\mathscr{D}}(\mathcal{F}(c), d) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(c, \mathcal{G}(d))$$

makes the below diagram commutes

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{D}}(\mathcal{F}(c), d) \arrow[r, "\Phi_{c,d}"] \arrow[d, "\mathcal{F}(f)^* \circ g_*"'] & \mathrm{Hom}_{\mathscr{C}}(c, \mathcal{G}(d)) \arrow[d, "f^* \circ \mathcal{G}(g)_*"] \\
\mathrm{Hom}_{\mathscr{D}}(\mathcal{F}(c'), d') \arrow[r, "\Phi_{c',d'}"] & \mathrm{Hom}_{\mathscr{C}}(c', \mathcal{G}(d'))
\end{tikzcd}$$

One should very trivial to verify this is actually the definition of adjoint functors where $\mathcal{F} \dashv \mathcal{G}$, that is to say, the Galois connection is actually the adjoint functors in preordered set category.

Proposition 4.3.2 ***Definition 4.1.7*** *and* ***Definition 4.3.1*** *are equivalent*

Proof This is trivial if you consider poset is a category. □

Once you formalize the definition, any examples we given under monotone Galois connection can be rephase into categorical language.
How? Once you consider our constructed A and B a category, then any (F, G) is pair of functors. To verfy Galois condition is to prove

$$\forall a \in A, b \in B, \ we\ have\ F(a) \le b \Longleftrightarrow a \le G(b)$$

in original definition, is equivalence to check the naturality bijection

$$\Phi_{a,b} : \mathrm{Hom}_{\mathscr{B}}(\mathcal{F}(a), b) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(a, \mathcal{G}(b))$$

in categorical language.
Now we introduce some other examples using categorical definition of Galois connection.

Example 4.3.3 Consider the categories $\mathscr{C} = \mathbf{Set}$ and $\mathscr{D} = \mathbf{Set}$ itself. Fix a set A and define functors:

$$\mathcal{F} : \mathbf{Set} \longrightarrow \mathbf{Set}, \quad \mathcal{F}(X) = X \times A,$$

$$\mathcal{G} : \mathbf{Set} \longrightarrow \mathbf{Set}, \quad \mathcal{G}(Y) = Y^A,$$

where Y^A denotes the set of all functions from A to Y.
For any sets X and Y, there is a natural bijection

$$\Phi_{X,Y} : \mathrm{Hom}_{\mathbf{Set}}(X \times A, Y) \xrightarrow{\simeq} \mathrm{Hom}_{\mathbf{Set}}(X, Y^A),$$

given by sending a map $f : X \times A \to Y$ to its curried form $\tilde{f} : X \to Y^A$, where $\tilde{f}(x)(a) = f(x, a)$. This is precisely the definition of the Cartesian closed structure of **Set**, and it makes the required naturality diagram commute. Hence $(\mathcal{F}, \mathcal{G})$ form a categorical Galois connection (an adjunction $\mathcal{F} \dashv \mathcal{G}$).

Example 4.3.4 As we remarked before, if we lift the definition of Galois connection to categorical language, then any adjoint functors $\mathcal{F} \dashv \mathcal{G}$ form a Galois connection. Let $\mathscr{C} = \mathbf{Set}$ and $\mathscr{D} = \mathrm{Vect}_{\mathbb{R}}$ be the category of vector space over the filed $\mathbb{R}$.
A classesical example of adjoint functors is the free-forgetful adjunction which free functor is given by $\mathcal{F} : \mathbf{Set} \longrightarrow \mathrm{Vect}_{\mathbb{R}}$:

$$\mathcal{F}(X) = \mathbb{R}^{(X)} = \oplus_{x \in X} \mathbb{R}$$
$$\mathcal{F}(f) \sum_{x \in X} a_x e_x = \sum_{x \in X} a_x e_{f(x)}$$

Where $\mathbb{R}^{(X)} := \{f : X \longrightarrow \mathbb{R} |\ finitely\ many\ x \in X\ s.t.\ f(x) \neq 0\}$. The free functor will add sufficient structure to make the set X a vector space over $\mathbb{R}$ and $f \in \mathrm{Hom}_{\mathbf{Set}}(X, Y)$. The forgetful functor is given by $\mathcal{G} : \mathrm{Vect}_{\mathbb{R}} \longrightarrow \mathbf{Set}$:

$$\mathcal{G}(V) = U(V)$$
$$\mathcal{G}(\varphi) = \varphi$$

The forgetful functor will simply forget the vector space structure and consider V as a set and any linear transformation $\varphi : V \longrightarrow W$ get forgeted to a set function. It is easy to verify these two

functors form an adjoint pair $\mathcal{F} \dashv \mathcal{G}$.
For andy set X and any vector space V, there is a natrual

$$\Phi_{X,V} : \mathrm{Hom}_{\mathrm{Vect}_{\mathbb{R}}}(\mathbb{R}^{(X)}, V) \xrightarrow{\simeq} \mathrm{Hom}_{\mathbf{Set}}(X, U(V))$$

Given a linear map $L : \mathbb{R}^{(X)} \longrightarrow V$, we can define $\Phi_{X,V}(L) : X \longrightarrow U(V)$ by $\Phi_{X,V}(L)(x) = L(e_x)$, where e_x is the basis vector corresponding to $x \in X$. Conversely, given a set map $\psi : X \longrightarrow U(V)$, we difine $\Phi^{-1}_{X,V}(\psi) : \mathbb{R}^{(X)} \longrightarrow V$ by the linear extension

$$\Phi^{-1}_{X,V}(\psi)\left(\sum_{x\in X} a_x e_x\right) = \sum_{x\in X} a_x \psi(x)$$

It is clear the map $\Phi_{X,V}$ is a bijection and natural in both X and V, thus $(\mathcal{F}, \mathcal{G})$ is a categorical Galois connection.

Galois connection can be considered as an universal property in many mathematical areas, any adjoint functors can be verified to form a Galois connection, or precisely they are the same thing.

Now we introduce a pretty example of Galois connection in the area intersect category and logic, which is a adjoint bridge between existential quantification and the categorical pullback.

Example 4.3.5 Let $\mathscr{C} = \mathbf{Set}$ and consider the slice category $\mathscr{D} = \mathbf{Set}/A$ for any fixed set A, which its object are pairs (X, f_X) where $X \in \mathrm{Ob}(\mathbf{Set})$ and $f_X \in \mathrm{Hom}_{\mathbf{Set}}(X, A)$. So any morphism $\phi : (X, f_X) \longrightarrow (Y, f_Y)$ is a function $\phi \in \mathrm{Hom}_{\mathbf{Set}}(X, Y)$ such that $f_Y \circ \phi = f_X$.
Now for fixed element $A, B \in \mathbf{Set}$ we have a funcion $p : A \longrightarrow B$ in $\mathrm{Hom}_{\mathbf{Set}}(A, B)$. We first define the pullback functor along p as

$$p^* : \mathbf{Set}/B \longrightarrow \mathbf{Set}/A, p^*(Y, f_Y) = (Y \times_B A, \pi_A)$$

where $Y \times_B A = \{(y, a) \in Y \times A | g(y) = p(a)\}$ is the pullback, and π_A is the projection to A. For another functor we have the Post-compostion with p as

$$\exists_p : \mathbf{Set}/A \longrightarrow \mathbf{Set}/B, \exists_p(X, f_X) = (X, p \circ f_X)$$

To verfy the Galois connection, for $(X, f) \in \mathbf{Set}/A$ and $(Y, g) \in \mathbf{Set}/B$, we need a natrual bijection

$$\mathrm{Hom}_{\mathbf{Set}/B}(\exists_p(X, f_X), (Y, g_Y)) \xrightarrow{\simeq} \mathrm{Hom}_{\mathbf{Set}/A}((X, f_X), p^*(Y, g_Y))$$

A morphism on the left is a function $\alpha : X \longrightarrow Y$ such that $g \circ \alpha = p \circ f$, where a morphism on the right is a function $\beta : X \longrightarrow Y \times_B A$ such that $\pi_A \circ \beta = f$ and $\pi_Y \circ \beta$ is a map $X \longrightarrow Y$ satisfying the same compatibility condition $g \circ \pi_Y \circ \beta = p \circ f$. Thus we can define the bijection as

$$\alpha \longmapsto (x \mapsto (\alpha(x), f(x)))$$
$$\beta \longmapsto \pi_Y \circ \beta$$

The condition $g(\alpha(x)) = p(f(x))$ guarantees that $(\alpha(x), f(x))$ indeed lies in the pullback $Y \times_B A$. And nutruality follows directly from the definition of the functors on morphisms. Thus we have the categorical Galois connection $(\exists_p, p^*)$.

Now we introduce our main definition, the categorical formalisation of antitone Galois connection.

Definition 4.3.6 (Categorical antitone Galois connection) For categories $\mathscr{C}$ and $\mathscr{D}$ with the functors $\mathcal{F} : \mathscr{C}^{\mathrm{op}} \longrightarrow \mathscr{D}$ and $\mathcal{G} : \mathscr{D} \longrightarrow \mathscr{C}^{\mathrm{op}}$. We say $\mathcal{F}$ and $\mathcal{G}$ form an antitone Galois connection if there exists a natural isomorphism

$$\Psi : \mathrm{Hom}_{\mathscr{D}}(\mathcal{F}(-), -) \longrightarrow \mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(-, \mathcal{G}(-))$$

which $\forall c \in \mathscr{C}^{\mathrm{op}}, d \in \mathscr{D}$, gives a bijection

$$\Psi_{c,d} : \mathrm{Hom}_{\mathscr{D}}(\mathcal{F}(c), d) \longrightarrow \mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(c, \mathcal{G}(d))$$

equivalently, a bijection

$$\Psi_{c,d} : \mathrm{Hom}_{\mathscr{D}}(\mathcal{F}(c), d) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(\mathcal{G}(d), c)$$

making the following diagram commute for all $f : c' \to c$ in $\mathscr{C}^{\mathrm{op}}$ (i.e., $f : c \to c'$ in $\mathscr{C}$) and $g : d \to d'$ in $\mathscr{D}$:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{D}}(\mathcal{F}(c), d) \arrow[r, "\Psi_{c,d}"] \arrow[d, "\mathcal{F}(f)^* \circ g_*"'] & \mathrm{Hom}_{\mathscr{C}}(\mathcal{G}(d), c) \arrow[d, "\mathcal{G}(g)^* \circ f_*"] \\
\mathrm{Hom}_{\mathscr{D}}(\mathcal{F}(c'), d') \arrow[r, "\Psi_{c',d'}"] & \mathrm{Hom}_{\mathscr{C}}(\mathcal{G}(d'), c')
\end{tikzcd}$$

Which is also a pair of adjoint functors, and we have to different way to consider it. On the one hand we can treat antitone Galois connection (F, G) as a pair of functors $(\mathcal{F}, \mathcal{G})$ and $\mathcal{F} \dashv \mathcal{G}$ but $\mathcal{F} : \mathscr{C}^{\mathrm{op}} \longrightarrow \mathscr{D}$ and $\mathcal{G} : \mathscr{D} \longrightarrow \mathscr{C}^{op}$, under this perspective we have bijection

$$\mathrm{Hom}_{\mathscr{D}}(\mathcal{F}(c), d) \cong \mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}} = \mathrm{Hom}_{\mathscr{C}}(\mathcal{G}(d), c)$$

On the other hand we can simply have $\mathcal{F} : \mathscr{C} \longrightarrow \mathscr{D}$ and $\mathcal{G} : \mathscr{D} \longrightarrow \mathscr{C}$ but require the naturality

$$\Phi_{c,d} : \mathrm{Hom}_{\mathscr{D}}(\mathcal{F}(c), d) \xrightarrow{\cong} \mathrm{Hom}_{\mathscr{C}}(\mathcal{G}(d), c)$$

Which those two perspectives are the same. In conclusion, the antitone Galois connection is still the adjoint functors in the world of category theory, but the adjoint with one category goes oppositly.
Now we have two biggest applications of antitone Galois connection, which is also our goal for this paper. The Galois theory in algebra and covering space theory in topology. Once we proved those two are the specific cases of our antitone Galois connection, then the bridge of Galois theory and covering space theory will be naturality built.

Example 4.3.7 (Galois theory) Anything introduced here required some basic knowledge of Galois theory. The goal of the example here only give some intuition about how antitone Galois connection becomes Galois theory after being applied in algebra. Details will be introduced in sections later.
To introduce Galois theory in categorical language, we have to meet two categories. For any known field extension L/k we have the category $\mathscr{C}$ where all objects are the intermediate field E of this extension such that $k \subseteq E \subseteq L$ and the morphism is the inclusion mapping $E_1 \hookrightarrow E_2$ when $E_1 \subseteq E_2$, and clear this is a poset. The second category $\mathscr{D}$ is the category of all subgroup of the Galois group $\mathrm{Gal}(L/k)$ with morphism as inclusion as well. In mathematical:

$$\mathrm{Ob}(\mathscr{C}) = \{E \ a \ field \mid k \subseteq E \subseteq L\}, \quad \mathrm{Hom}_{\mathscr{C}}(E_1, E_2) : E_1 \hookrightarrow E_2 \ when \ E_1 \subseteq E_2$$
$$\mathrm{Ob}(\mathscr{D}) = \{H \ a \ group \ |\{e\} \leq H \leq \mathrm{Gal}(L/k)\}, \quad \mathrm{Hom}_{\mathscr{D}}(H_1, H_2) : H_1 \hookrightarrow H_2 \ when \ H_1 \leq H_2$$

If you know some Galois theory then you can understand the identity group $\{e\}$ in the object is not meaningless, since it is the group which fixes not only k but L.
We claim that there is an antitone Galois connection between $\mathscr{C}$ and $\mathscr{D}$, so there should be a pair of adjoint functors between those categories with one goes opposite but which one does not matter. To meet calssical Galois theroy, we say we have $\mathcal{F} : \mathscr{C} \longrightarrow \mathscr{D}^{\mathrm{op}}$ and $\mathcal{G} : \mathscr{D}^{\mathrm{op}} \longrightarrow \mathscr{C}$ defined on any intermediate field $E \in \mathrm{Ob}(\mathscr{C})$ and subgroup $H \in \mathrm{Ob}(\mathscr{D})$ as

$$\mathcal{F}(E) = \mathrm{Gal}(L/E)$$
$$\mathcal{G}(H) = L^H := \{x \in L \mid \sigma(x) = x, \forall \sigma \in H\}$$

And what Galois theory says is that this gives an antitone Galois connection, or in categorical language we have a natural isomorphism

$$\Psi_{E,H} : \mathrm{Hom}_{\mathscr{D}^{\mathrm{op}}}(\mathcal{F}(E), H) \xrightarrow{\cong} \mathrm{Hom}_{\mathscr{C}}(E, \mathcal{G}(H))$$

We will talk about this detaily later, after we introduce the Galois theory. We will understand all the concepts of Galois theory is trying to prove this natural isomorphism brick by brick, not categorically but algebrically.

Example 4.3.8 (Classification theorem of covering space) As before, anything introduced here required some basic knowledge of algebraic topology. The goal of the example here only give some intuition about how antitone Galois connection becomes covering space theory after being applied in topology. Details will be introduced in sections later.
Before we give out our two categories we have some set up. Everything is working on a pointed topological space (X, x_0) that is:
(i) Path-connected: $\forall x, y \in X$, there is a continous function $\gamma : [0,1] \longrightarrow X$ with $\gamma(0) = x$ and $\gamma(1) = y$.
(ii) Locally path-connected: $\forall x \in X$ has a neighborhood basis consisting of path-connected sets.
(iii) Semilocally simply connected: $\forall x \in X$ has a neighborhood U such that the homomorphism $\pi_1(U, x) \hookrightarrow \pi_1(X, x)$ induced by inclusion is trivial.
Moreover we have a universal cover $p : (\tilde{X}.\tilde{x_0}) \longrightarrow (X, x_0)$. Then we have our category $\mathscr{C}$ contains all connected covers as objects. Here connected cover means a topological space which is path-connected itself, which can not factors into a disjoint union of two non-empty open sets. So for $p_1 : (Y_1, y_1) \longrightarrow (X, x_0), p_2 : (Y_2, y_2) \longrightarrow (X, x_0) \in \mathrm{Ob}(\mathscr{C})$ we have $f \in \mathrm{Hom}_{\mathscr{C}}(p_1, p_2)$ such that $p_2 \circ f = p_1$ and $f(y_1) = y_2$ preserves the based point, and it is trivial to verify f is a cover itself. Consider another category $\mathscr{D}$ whose objects are the subgroups of $\pi_1(X, x_0)$ and the morphisms is simply the inclusion as Galois theory. In mathematical we have

$$\mathrm{Ob}(\mathscr{C}) = \{p \textit{ a cover} \mid p : (Y, y_0) \longrightarrow (X, x_0)\}, \quad \mathrm{Hom}_{\mathscr{C}}(p_1, p_2) = \{f \textit{ a cover} \mid f : (Y_1, y_1) \longrightarrow (Y_2, y_2)\}$$
$$\mathrm{Ob}(\mathscr{D}) = \{H \textit{ a group} \mid H \leq \pi_1(X, x_0)\}, \quad \mathrm{Hom}_{\mathscr{D}}(H_1, H_2) : H_1 \hookrightarrow H_2 \textit{ when } H_1 \leq H_2$$

Like before, covering space theory is a antitone Galois connection that we have $\mathcal{F} : \mathscr{C} \longrightarrow \mathscr{D}^{\mathrm{op}}$ and $\mathcal{G} : \mathscr{D}^{\mathrm{op}} \longrightarrow \mathscr{C}$ as adjoint functors that for any connected cover $p : Y \longrightarrow X$ and any subgroup H we have

$$\mathcal{F}(p : (Y, y_0) \longrightarrow (X, x_0)) = p_*(\pi_1(Y, y_0))$$
$$\mathcal{G}(H) = p_H : \tilde{X}/H \longrightarrow X$$

where p_H is the covering space obtained by quotienting the universal cover by the action of subgroup H. One should verify this gives us an antitone Galois connection by the natural isomorphism

$$\Psi_{Y,H} : \mathrm{Hom}_{\mathscr{D}^{\mathrm{op}}}(\mathcal{F}(Y), H) \xrightarrow{\cong} \mathrm{Hom}_{\mathscr{C}}(Y, \mathscr{G}(H))$$

And this is exactly what classification theory trying to say in categorical language. We should come back later after we introduce covering sapce formally.

To reach our primry goal, we should introduce some important properties of adjoint functors or Galois connection. The first one is the fixed-point equivalence which gives us an equivalence between two subcategories defined by the unit and counit of the adjoint functors.

Proposition 4.3.9 *(Fixed-point equivalence) Let $\mathcal{F} : \mathscr{C} \longrightarrow \mathscr{D}$ and $\mathcal{G} : \mathscr{D} \longrightarrow \mathscr{C}$ be adjoint functors $\mathcal{F} \dashv \mathcal{G}$, with unit $\eta : \mathrm{id}_{\mathscr{C}} \Rightarrow \mathcal{G}\mathcal{F}$ and counit $\varepsilon : \mathcal{F}\mathcal{G} \Rightarrow \mathrm{id}_{\mathscr{D}}$. Define the* fixed-point subcategories

$$\mathscr{C}_{\mathrm{fix}} := \{c \in \mathscr{C} \mid \eta_c \textit{ is an isomorphism}\}, \quad \mathscr{D}_{\mathrm{fix}} := \{d \in \mathscr{D} \mid \varepsilon_d \textit{ is an isomorphism}\}.$$

Then the adjoint pair $(\mathcal{F}, \mathcal{G})$ restricts to an equivalence of categories

$$\mathcal{F}|_{\mathscr{C}_{\mathrm{fix}}} : \mathscr{C}_{\mathrm{fix}} \xrightarrow{\simeq} \mathscr{D}_{\mathrm{fix}}, \qquad \mathcal{G}|_{\mathscr{D}_{\mathrm{fix}}} : \mathscr{D}_{\mathrm{fix}} \xrightarrow{\simeq} \mathscr{C}_{\mathrm{fix}}.$$

Chapter 5

Categorical Galois Theory

Galois theroy is one of the most powerful algebric theory. Consider the quadratic equation $ax^2 + bx + c = 0$, we say $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ is the formula of solution, namely the radical. For any higher n degree polynomial equation $\sum_{i=0}^{n} a_i x^i = 0$, the radical to this equation is a family of functions $x_i = f_i(a_0, a_1, ...a_n)$ which satisfies the equations, where any functions only required the operations of $+, -, \times, \div, \sqrt{}$. The biggest contribution of Galois Theory is the proof of the non-existence of the radical for any equations with degree equal or bigger than 5. In this chapter we will introduce some basic techique of Galois theory and see how it is connected to covering space.
The genius idea of Galois is to connect every equations to its symmetric group. More precisely, every equations's spliting field has a Galois groups and the existence of radical is related to the Galois group is solvable or not.
All our work will generalize Galois theory as categorical language. For a formal discription of Galois theory I suggest reader visit [3, Expository papers website of Mr Keith Conrad], which provide a comprehensive Galois theory and also many other expository writing of different mathematical fields. Our journey begin with field extensions.

5.1 Field extensions

Definition 5.1.1 (Categories Fields) Let **Ring** denote the category whose objects are rings and whose morphisms are ring homomorphisms. We say a ring F is a field if $(F \setminus \{0\}, \times)$ is a commutative group. Thus, the category **Field** is the full subcategory of **Ring** consisting of fields. Equivalently, **Field** is the category where:

$$\mathrm{Ob}(\mathbf{Field}) = \{F \in \mathrm{Ob}(\mathbf{Ring}) | (F \setminus \{0\}, \times) \textit{ is commutative group}\}$$
$$\mathrm{Hom}_{\mathbf{Field}}(F_1, F_2) = \mathrm{Hom}_{\mathbf{Ring}}(F_1, F_2)$$

Where any morphism in **Field** is a mapping induced by a ring homomorphism.

The definition is not new to us, fields play very provital role in Galois theory, where Galois theory build a bridge between field extension and group theory.

Definition 5.1.2 (Field Extension) A field extension of a fixed field F is defined as the co-slice category $F/\mathbf{Field}$, whose objects and morphisms are given by:

$$\mathrm{Ob}(F/\mathbf{Field}) = \{(L, i) \mid L \in \mathrm{Ob}(\mathbf{Field}),\ i \in \mathrm{Hom}_{\mathbf{Field}}(F, L)\};$$
$$\mathrm{Hom}_{F/\mathbf{Field}}((L_1, i_1), (L_2, i_2)) = \{g \in \mathrm{Hom}_{\mathbf{Field}}(L_1, L_2) \mid g \circ i_1 = i_2\}.$$

Since the kernel of any field homomorphism is an ideal and a field has only the trivial ideals $\{0\}$ and itself, every $i : F \to L$ is injective. Consequently, i embeds F isomorphically onto its image $i(F) \subseteq L$.

In practice, one often identifies F with its image $i(F)$ and views the extension as an inclusion $F \subseteq L$, denoted L/F. However, in the categorical setting we retain the explicit homomorphism i as part of the data.

We have a functor to represent how big our extension is.

Definition 5.1.3 (Degree of an Extension) Let $(L, i) \in \mathrm{Ob}(F/\mathbf{Field})$. Through i, we can regard L as an F-vector space via the action $a \cdot x := i(a)x \quad (a \in F,\ x \in L)$. If this vector space is finite-dimensional, we define the degree of the extension as $[L : F] := \dim_F L$. This assignment extends to a functor:

$$[\,\cdot\,/F] : (F/\mathbf{Field})^{\mathrm{fin}} \longrightarrow (\mathbb{N}, |),$$

where $(\mathbb{N}, |)$ is the poset category of natural numbers ordered by divisibility.

Now we provide some basic example of field extensions.

Example 5.1.4 For any field F, the pair (F, id_F) is an object of $F/\mathbf{Field}$ as the trivial extension F/F. Moreover, it is the initial object of this category that for every object (L, i) there exists exactly one morphism

$$\mathrm{id}_F : (F, \mathrm{id}_F) \longrightarrow (L, i),$$

namely the map id_F itself which satisfies $i \circ \mathrm{id}_F = i$. This corresponds to the fact that F is contained in every extension L in the unique way prescribed by $i : F \hookrightarrow L$.

Example 5.1.5 Consider $F = \mathbb{R}$. The inclusion $f : \mathbb{R} \hookrightarrow \mathbb{C}$ makes $(\mathbb{C}, i)$ an object of $\mathbb{R}/\mathbf{Field}$. The field $\mathbb{C}$ can be considered as a two-dimensional $\mathbb{R}$-vector space with the basis $\{1, i\}$. Hence the degree functor gives $[\mathbb{C} : \mathbb{R}] = 2$. The automorphism group of $(\mathbb{C}, f)$ in $\mathbb{R}/\mathbf{Field}$ consists of those field automorphisms of $\mathbb{C}$ that fix $\mathbb{R}$ pointwise. The only non-trivial such automorphism is complex conjugation, so

$$\mathrm{Aut}_{\mathbb{R}/\mathbf{Field}}(\mathbb{C}, i) \cong \mathbb{Z}/2\mathbb{Z}.$$

Here we use f as our inclusion mapping to avoid the misunderstanding of imaginary unit i.

Example 5.1.6 Let $F = \mathbb{Q}$. Define $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ with the usual operations. The canonical inclusion $i : \mathbb{Q} \hookrightarrow \mathbb{Q}(\sqrt{2})$ gives the object $(\mathbb{Q}(\sqrt{2}), i)$ in $\mathbb{Q}/\mathbf{Field}$. Again $\mathbb{Q}(\sqrt{2})$ is a two-dimensional $\mathbb{Q}$-vector space with basis $\{1, \sqrt{2}\}$, so $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$. There is a non-trivial field automorphism $\sigma : \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{2})$ sending $\sqrt{2} \mapsto -\sqrt{2}$. However, σ is *not* an automorphism of the object $(\mathbb{Q}(\sqrt{2}), i)$ because $\sigma \circ i = i$ (not a different embedding). Hence

$$\mathrm{Aut}_{\mathbb{Q}/\mathbf{Field}}(\mathbb{Q}(\sqrt{2}), i) = \{\mathrm{id}\}.$$

Example 5.1.7 Take $F = \mathbb{Q}$ and let $L = \mathbb{Q}(\sqrt{2})$ as above. Consider two distinct homomorphisms

$$i_1 : \mathbb{Q} \hookrightarrow L, \quad i_1(a) = a, \qquad i_2 : \mathbb{Q} \hookrightarrow L, \quad i_2(a) = a \text{ (but now } \sqrt{2} \text{ is interpreted as } -\sqrt{2}).$$

Then (L, i_1) and (L, i_2) are two *different* objects of $\mathbb{Q}/\mathbf{Field}$, because the structure maps i_1, i_2 differ. They are isomorphic as fields, but not as extensions: there is no morphism $\phi : (L, i_1) \to (L, i_2)$ satisfying $\phi \circ i_1 = i_2$, because any field homomorphism $\phi : L \to L$ must either be the identity or $\sigma : \sqrt{2} \mapsto -\sqrt{2}$, and one checks that neither satisfies the required commutativity with i_1, i_2.
This illustrates that an object of $F/\mathbf{Field}$ is not merely a field L, but a field together with a specific way of viewing F inside it.

Example 5.1.8 Let $F = \mathbb{Q}$. We have objects $(\mathbb{Q}(\sqrt{2}), j)$ and $(\mathbb{Q}(\sqrt{2}, \sqrt{3}), i)$ where $j : \mathbb{Q} \hookrightarrow \mathbb{Q}(\sqrt{2})$ and $i : \mathbb{Q} \hookrightarrow \mathbb{Q}(\sqrt{2}, \sqrt{3})$ are the natural inclusions. The inclusion map $\phi : \mathbb{Q}(\sqrt{2}) \hookrightarrow \mathbb{Q}(\sqrt{2}, \sqrt{3})$ satisfies $\phi \circ j = i$, hence it is a morphism

$$\phi : (\mathbb{Q}(\sqrt{2}), j) \longrightarrow (\mathbb{Q}(\sqrt{2}, \sqrt{3}), i)$$

in $\mathbb{Q}/\mathbf{Field}$. This morphism reflects the classical fact $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\sqrt{2}, \sqrt{3})$.

Example 5.1.9 Consider the objects $(\mathbb{Q}(\sqrt{2}), i_1)$ and $(\mathbb{Q}(\sqrt{3}), i_2)$ in $\mathbb{Q}/\mathbf{Field}$. There exists *no* field homomorphism $\phi : \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{3})$ with $\phi \circ i_1 = i_2$ (unless $\sqrt{2}$ is mapped to an element whose square is 3, which is impossible inside $\mathbb{Q}(\sqrt{3})$). Therefore

$$\mathrm{Hom}_{\mathbb{Q}/\mathbf{Field}}((\mathbb{Q}(\sqrt{2}), i_1), (\mathbb{Q}(\sqrt{3}), i_2)) = \varnothing.$$

This shows that the existence of a morphism in $F/\mathbf{Field}$ is a non-trivial condition; it precisely means that the target extension "contains" the source extension in a way compatible with the chosen embeddings of F.

These examples demonstrate how the categorical definition $F/\mathbf{Field}$ captures the familiar notions of field extensions, inclusions, degrees, and automorphisms, while making the role of the base-field embedding explicit.

Proposition 5.1.10 *Let* $(K, i_1 : F \hookrightarrow K)$ *and* $(L, i_2 : K \hookrightarrow L)$ *in* $\mathrm{Ob}(F/\mathbf{Field})$ *and* $\mathrm{Ob}(K/\mathbf{Field})$ *respectively. Then* $(L, i_2 \circ i_1) \in \mathrm{Ob}(F/\mathbf{Field})$*, and the degree functors satisfy*

$$[L : F] = [K : F] \cdot [L : K],$$

where $[L : F]$ *is computed via the* F*-vector space structure induced by* $i_2 \circ i_1$*, with* $[K : F]$ *via* i_1 *and* $[L : K]$ *via* i_2*.*

Proof The composition $i_2 \circ i_1$ makes L an F-algebra. As an F-vector space,

$$L \cong K \otimes_F L \quad \text{(as } K\text{-modules)},$$

but more precisely, choosing bases: if $\{e_1, \ldots, e_n\}$ is an F-basis of K and $\{f_1, \ldots, f_m\}$ a K-basis of L, then $\{e_i f_j\}$ is an F-basis of L. Hence $\dim_F L = \dim_F K \cdot \dim_K L$. □

After those basic set up, now we have to restrict our extension. In Galois theory we do not study field extension randomly, we mainly focus on algebraic extension. In classical textbook you may have already studied the original definition, We say L/F is algebraic if every $\alpha \in L$ is an algebraic element of F or algebraic over F written as α is alg/F. And to say $\alpha \in L$ is alg/F if $\exists f \in F[x] \backslash \{0\}$ *s.t.* $f(\alpha) = 0$. Now we provide a more categorical perspective of this definition which based on F-algebra. A F-algebra is a ring homomorphism $\eta : F \longrightarrow A$ where F is considered as a ring and A is any ring. Equivalently it is an object in the cosline category $F/\mathbf{Ring}$, and such η induced an F-linear structure on A via $\lambda \cdot a := \eta(\lambda)a$ for $\lambda \in F$, $a \in A$.
In our case, $F[x]$ together with $i : F \hookrightarrow F[x]$ becomes the free F-algebra on one generator, that is, for any F-algebra (A, η) and $\forall a \in A$, $\exists!$ F-algebra homomorphism $\alpha : F[x] \longrightarrow A$ such that $\alpha(x) = a$. And the universal property gives a bijection as

$$\mathrm{Hom}_{F/\mathbf{Ring}}(F[x], A) \cong A$$

which between F-algebra homomorphism from $F[x]$ to A and the elements of underlying set of A. Now our definition of algebraic extension can be defined on element $\alpha \in L$ which can be considered as ring homomorphism, which gives us a categorical formalization of algebraic extension.

Definition 5.1.11 (Algebraic Extension) Let $(L, i : F \hookrightarrow L)$ be an object of the category $F/\mathbf{Field}$. An F-algebraic point of L is an F-algebra homomorphism $\alpha : F[x] \to L$ whose kernel is a non-zero ideal of $F[x]$, and we say α is algebraic over F denoted as alg/F.
The extension (L, i) is called *algebraic over* F if every F-algebra homomorphism $\alpha : F[x] \to L$ is an algebraic point. Equivalently, every element of L (in the classical sense) is algebraic over F. An extension that is not algebraic is called *transcendental*.

The definition above bridges classical and categorical viewpoints. Classically, we say $\alpha \in L$ is algebraic if there exists a non-zero polynomial $f(x) \in F[x]$ with $f(\alpha) = 0$. Categorically, this is equivalent to say the evaluation homomorphism $\mathrm{ev}_\alpha : F[x] \to L$, $f(x) \mapsto f(\alpha)$ has non-zero kernel. The extension (L, i) is algebraic precisely when the canonical map

$$\coprod_{\alpha \in L} F[x] \longrightarrow L$$

where the coproduct is in the category of F-algebras has the property that each component ev_α factors through a quotient $F[x]/(p_\alpha(x))$.

Proposition 5.1.12 $[L : F] < \infty \Longrightarrow (L, i)$ *is algebraic.*

Proof For $[L : F] = n < \infty$, L/F is a n-dimensional vector space over field F. That is $\forall \alpha \in L$, $\{\alpha^0, \alpha^1, \alpha^2, \ldots, \alpha^n\}$ a linear dependent basis over F. That is to say, $\exists \{k_0, k_1, \ldots, k_n\}$ such that $\sum_{i=0}^n k_i \alpha^i = f(\alpha) = 0$ where $f \in k[x] \setminus \{0\}$. $\square$

But an algebraic extension is far from the approach we will use to discover the symmetric in field extension.

Definition 5.1.13 (Minimal Polynomial) Let $(L, i) \in F/\mathbf{Field}$ and let $\alpha : F[x] \to L$ be an algebraic point. Since $F[x]$ is a principal ideal domain, $\ker(\alpha)$ is generated by a unique monic polynomial $p_\alpha(x) \in F[x]$. This polynomial is called the *minimal polynomial of α over F*.
In categorical terms, $p_\alpha(x)$ is the generator of the kernel of the F-algebra homomorphism α, and we have a factorization:

$$\begin{tikzcd} F[x] \arrow[r, "\alpha"] \arrow[d, "\pi"'] & L \\ F[x]/(p_\alpha(x)) \arrow[ur, "\tilde{\alpha}"'] & \end{tikzcd}$$

where π is the quotient map and $\tilde{\alpha}$ is an injective F-algebra homomorphism (hence a field embedding).

Definition 5.1.14 (Simple Extension) Let $(L, i) \in F/\mathbf{Field}$ and $\alpha : F[x] \to L$ an algebraic point with minimal polynomial $p_\alpha(x)$. The *simple extension generated by* α is the object $(F(\alpha), j) \in F/\mathbf{Field}$ defined as:

$$F(\alpha) := F[x]/(p_\alpha(x)), \quad j : F \hookrightarrow F(\alpha) \text{ the natural inclusion.}$$

There is a canonical morphism $\tilde{\alpha} : (F(\alpha), j) \to (L, i)$ in $F/\mathbf{Field}$ induced by α, which is injective. Thus $(F(\alpha), j)$ is (isomorphic to) the smallest subextension of (L, i) containing $\alpha(x)$.

Proposition 5.1.15 *(Algebraic Extensions as Unions of Simple Extensions) Let (L, i) be an algebraic extension of F. Then (L, i) is isomorphic to a filtered colimit of simple extensions in $F/\mathbf{Field}$:*

$$(L, i) \cong \varinjlim_{\alpha \in \Lambda} (F(\alpha), j_\alpha),$$

where Λ is the set of all algebraic points $\alpha : F[x] \to L$.

Theorem 5.1.16 *(Primitive Element Theorem) Let (L, i) be a finite separable extension of F (separability will be defined later). Then there exists an algebraic point $\alpha : F[x] \to L$ such that*

$$(L, i) \cong (F(\alpha), j) \quad \textit{in } F/\mathbf{Field}.$$

That is, every finite separable extension is simple.

Chapter 6

Covering Space

6.1 Topological covering Space

We have already talked about the fundamental group and fundamental groupoid for topological space, then comes up with van Kampen theorem to calculate the fundamental structure for some nontrivial space. What we more focus on is the equivalent description to van Kampen theorem, show us the power of categorical language also give us some intuition to homotopy theory.
Now we come to covering space, recall how we prove $\pi_1(S^1) = \mathbb{Z}$ by covering the real line $\mathbb{R}$ circle by circle to S^1. But covering space is more than that to compute fundamental group. Unlike any other textbook, we will see covering space in categorical language and in homotopic way.

Definition 6.1.1 (Evenly Covering Space) An evenly covering space for space X consists a pair $(\hat{X}, p)$ such that $p : \hat{X} \longrightarrow X$ is surjective and $\forall x \in X$, there is an open neighborhood U_x where

$$p^{-1}(U_x) = \bigsqcup_{i \in I} V_i$$

for some disjoint open set V_i in $\hat{X}$, and $p|_{V_i} : V_i \longrightarrow U_x$ is homeomorphism for every $i \in I$. We call $\{V_i\}$ is the sheets of $\hat{X}$ over X and $F_x = p^{-1}(x) \in \bigsqcup_{i \in I} V_i$ is a fiber of the covering p.

It should be mentioned that the difference between evenly covering space and covering space is that the preimage of any neighborhood can be written as disjoint union or not. But in Algebric Topological the covering we describe are almost all evenly, so in later words we consider cover is an evenly cover. One should verify any homomorphism is a cover. Most classical example is the covering we used to show $\pi_1(S^1) = \mathbb{Z}$ by $p = \mathbb{R} \longrightarrow S^1$. We will see the strong connection between covering space and the fundamental groupoid or group, the following theorem describes how p acts on the space.

Theorem 6.1.2 *(Unique path lifting theorem) For a cover $p : \hat{X} \longrightarrow X$ with $e_0 \in F_{x_0}$ where $x_0 \in X$, we have*

$$\forall f \in \{f \ a \ path : I \longrightarrow X | f(0) = x_0\},$$
$$\exists! g \in \{g \ a \ path : I \longrightarrow \hat{X} | g(0) = e_0\}$$

such that $p \circ g = f$

Proof For a evenly covered base space X, if f lies entirely on the sheet $S_{x_0} = \{V_i\}$ with respect to U_{x_0}, then $g = q \circ f$ where q is the homomorphic inverse of p such that $q|_{V_i} \circ p|_{V_i} = id|V_i, \forall V_i \in S_{x_0}$, and the uniqueness comes from the unique inverse for homeomorphism. For a general case, by Lebesgue's Covering Lemma I is compact that we can partition I by $0 = t_0 < t_1 < ... < t_n = 1$ according to the covering, making f maps every closed $[t_i, t_{t+1}]$ to an evenly covered neighborhood of $f(t_i)$, then by the evenly cover case and induction we can obatin the unique lifting. $\square$

What the unique path lifting theorem means is that for the cover $p : \hat{X} \longrightarrow X$, any fundamental points x_0 in based space X lifted into the fiber F_{x_0}, and $\forall e_i \in F_{x_0}$. This theorem give us the intuition that the lifting also act on the homotopic calss of each path.

Corollary 6.1.3 *For a covering $p : \hat{X} \longrightarrow X$ and a homotopy in X that $h : f \simeq f'$ start from x_0 lifts uniquely to a homotopy $H : g \simeq g'$ in $\hat{X}$ start from e_0.*

Proof For the homotopy $h : I \times I \longrightarrow X$, by Lebesgue's Covering Lemma we can subdivide the compact $I \times I$ into many subsquares each of which maps to a fundamental neighborhood of f, by **Theorem 6.1.2** we see any paths in the homotopic class lifts to a unique path that h lifts uniquely to a homotopy $H : I \times I \longrightarrow \hat{X}$ where f and f' lift to g and g' with $H : g \simeq g'$ that $H(0, 0) = e_0$, and one should be easily verified that $h = p \circ H$. $\square$

Proposition 6.1.4 *The induced group homomorphism $p_* : \pi_1(\hat{X}, e_i) \longrightarrow \pi_1(X, x_0)$ is a monomorphism, for all $e_i \in F_{x_0}$.*

Before discussing the algebric structure of covering space, we first define two important covering space according to special group structures.

Definition 6.1.5 Consider the covering $p : \hat{X} \longrightarrow X$. We say p is regular if $\pi_1(X, x_0)/p_*(\pi_1(\hat{X}, e_i))$ is still a group, p is universal if $\pi_1(X, x_0)/p_*(\pi_1(\hat{X}, e_i)) = \pi_1(X, x_0)$ and it is the normal subgroup of $\mathbb{Z}$ so it's regular.

Example 6.1.6 Any integer multiple winding of S^1 is a normal cover, say if a covering wrap the S^1 three times it is easy to calculate $p_*(\hat{X}, e_i) = 3\mathbb{Z}$. Moreover, the covering $p : \mathbb{R} \longrightarrow S^1$ is universal because $\mathbb{R}$ is simply connected that $\pi_1(\mathbb{R}, e_i) = 0$, we also call $\mathbb{R}$ is the universal cover of S^1.

The relationship between covering space $\hat{X}$ and based space X is defined on the structure of the underlying fundamental groups. But we want to study the structure more specifically, says we should generalize the idea of covering from topological space to any algebric objects. One way to do this is to define the covering space on categories. Once we have the categorical covering, the induced homomorphism p_* will become a covering contain more information rather than a mapping. The way to generalize the covering space is working on the category called morphism category.

Definition 6.1.7 (Morphisms Category) Let $\mathscr{C}$ be any category and a $x \in \mathrm{ob}(\mathscr{C})$, the morphism category of $\mathscr{C}$ under x is a category $x\backslash\mathscr{C}$ such that

$$\mathrm{Ob}(x\backslash\mathscr{C}) := \bigcup_{y_i \in \mathrm{Ob}(\mathscr{C})} \mathrm{Hom}_{\mathscr{C}}(x, y_i)$$

which is all the morphisms in $\mathscr{C}$ that from x to others. For any $f, g \in \mathrm{Ob}(x\backslash\mathscr{C})$, $f : x \longrightarrow y_1$, $g : x \longrightarrow .y_2$ for some $y_1, y_2 \in \mathrm{Ob}(\mathscr{C})$, then the morphism in $x\backslash\mathscr{C}$ is defined as the compostion

$$\mathrm{Hom}_{x\backslash\mathscr{C}}(f, g) := \mathrm{Hom}_{\mathscr{C}}(y_1, y_2)$$

which means the morphisms $\gamma : f \longrightarrow g$ are the morphisms $\gamma : y_1 \longrightarrow y_2$ such that $\gamma(f) := \gamma \circ f = g$ makes the diagram below commutes

$$\begin{tikzcd}
 & x \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, phantom, "\xrightarrow[\gamma\in\mathrm{Hom}_{x\backslash\mathscr{C}}(f,g)]{}"] & \\
y_1 \arrow[rr, "\gamma\in\mathrm{Hom}_{\mathscr{C}}(y_1,y_2)"'] & & y_2
\end{tikzcd}$$

The motivation of defining this is that we want to generalize our covering idea to fundamental group or groupoid, as you can see both of them can be considered as categorical objects, so the covering should be a functor.

Definition 6.1.8 (Covering of Groupoids) Let $\mathscr{B}$ and $\mathscr{C}$ be two small connected category, meaning the objects form a set not a proper class, and any two object have at least one invertalbe morphism between them. Then the covering between $\mathscr{B}$ and $\mathscr{C}$ is a surjective functor $p : \mathscr{B} \longrightarrow \mathscr{C}$ such that

$$p : \mathrm{Ob}(b\backslash\mathscr{B}) \longrightarrow \mathrm{Ob}(p(b)\backslash\mathscr{C})$$

is a bijection $\forall b \in \mathscr{B}$. Similar to the covering of topological space, for a $c \in \mathscr{C}$, the fiber of c is the set $F_c = \{b \in \mathscr{B} | p(b) = c\}$ and $\forall c \in \mathscr{C}$ we have

$$p^{-1}(\mathrm{Ob}(c\backslash\mathscr{C})) = \bigsqcup_{b_i \in F_c} \mathrm{Ob}(b_i\backslash\mathscr{B})$$

The objects in $x\backslash\mathscr{C}$ is a set of morphisms of $\mathscr{C}$ with source x we denote is as $\mathrm{St}_{\mathscr{C}}(x)$, so the covering restricts to the bijection $p : \mathrm{St}_{\mathscr{B}}(b) \longrightarrow \mathrm{St}_{\mathscr{C}}(p(b))$ and $p^{-1}(\mathrm{St}_{\mathscr{C}}(c)) = \bigsqcup_{b_i \in F_c} \mathrm{St}_{\mathscr{B}}(b_i)$.

The definition of covering of groupoids is strictly followed by the covering of topological space, with the based point x_0 changed to the morphism with source x_0. We have the statement below naturally.

Proposition 6.1.9 *The induced functor $\prod(p) : \prod(\hat{X}) \longrightarrow \prod(X)$ is a covering of groupoid if $p : \hat{X} \longrightarrow X$ is a covering of topological space.*

Proof This is actually the same statement of **Theorem 6.1.2** and **Corollary 6.1.3** as long as consider any path is a invertible morphism in the groupoid as a category. □

The key of algebric topology is to use algebric structure to study and verfy difference topological space, and our main work is also trying to discover any potential example to build a bridge between difference mathematical field. We will see how covering space and Galois theory connected to each other by the underlying group of topological space through covering space. Now we will study more on the algebric side of covering space.

Definition 6.1.10 (Automorphism group of groupoid) Let $\mathscr{C}$ be any groupoid, the automorphism of $x \in \mathscr{C}$ are all the objects in $x\backslash\mathscr{C}$ which from x sends to x itself and denoted as $\pi(\mathscr{C}, x)$, in mathematical words

$$\pi(\mathscr{C}, x) := \mathrm{Hom}_{\mathscr{C}}(x, x)$$

It is obvious that if the $\mathscr{C}$ is any fundamental groupoid of a topological space then the automorphism group is actually the fundamental group with respect to the chosen x_0, that is

$$\pi(\prod(X), x_0) = \mathrm{Hom}_{\prod(X)}(x_0, x_0) = \pi_1(X, x_0)$$

So for any covering space $p : \hat{X} \longrightarrow X$ we have the covering of underlying groupoid with the connected of groups, the covering of groupoid indicate some information on groups.

Proposition 6.1.11 *Let $p : \hat{X} \longrightarrow X$ be a covering of topological space, then the induced morphism $p_* : \pi(\prod(\hat{X}), e_i) \longrightarrow \pi(\prod(X), x_0)$ is a monomorphism, where $x_0 \in X$ is the based point and $e_i \in F_{x_0} = \{e_i\}_{i \in I}$. Moreover, $\forall e_j, e_k \in F_{x_0}$, $p_*(\pi(\prod(\hat{X}), e_j))$ is conjugate to $p_*(\pi(\prod(\hat{X}), e_k))$ in $\pi(\prod(X), x_0)$.*

Proof The injectivity is trivial by the bijection of $\prod(p)$ on $\mathrm{St}(e_i)$. $\prod(\hat{X})$ is a groupoid so there is a path $g : e_j \longrightarrow e_k$, the conjugation is given by the $p(g) \in \pi(\prod(X), x_0)$, that is

$$p_*(\pi(\prod(\hat{X}), e_k)) = [p_*(g)] \circ [p_*(\pi(\prod(\hat{X}), e_j))] \circ [p_*(g)]^{-1}$$ □

The proposition is saying the underlying group of based space at x_0 is hiden in any copy of group at $e_i \in F_{x_0}$, when the covering space covers to X by p, all the group structure on F_{x_0} will onto the one group at x_0.

Proposition 6.1.12 *The group* $p_*(\pi(\prod(\hat{X}), e_j))$ *runs through all conjugates of* $p_*(\pi(\prod(\hat{X}), e_i)) \in \pi(\prod(X), x_0)$ *as* e_j *runs through* F_{x_0}.

Proof This is trivial that the surjectivity of the groupoid covering functor $\prod(p)$ on $\mathrm{St}(e_i)$ and apply the same method in **Proof 5.3** □

Now the connection between covering space of topological space and groupoid, together with the underlying fundamental group is very clear as the commutative diagram shows below, we consider the fundamental group of a space as the automorphism group under the groupoid category at specific object, all our work is trying to build a categorical eyesight of algebric topology.

$$\begin{tikzcd}
\hat{X} \arrow[r, "\prod"] \arrow[d, "p"] & \prod(\hat{X}) \arrow[r, "\pi(\mathscr{C},x)"] \arrow[d, "\prod(p)"] & \pi(\prod(\hat{X}), e_i) \arrow[r, equal] \arrow[d, "p_*"] & \pi_1(\hat{X}, e_i) \arrow[d, "p^*"] \\
X \arrow[r, "underlying\ groupoid"'] & \prod(X) \arrow[r, "automorphism\ group"'] & \pi(\prod(X), x_0) \arrow[r, equal] & \pi_1(X, x_0)
\end{tikzcd}$$

It is clear that the fundamental group of covering space should have a same corresponding relation as groupoid, if we consider the group as the automorphism of the groupoid category under the covering, then there should be some algebric relation between fundamental groups.

Chapter 7

Basic homotopy theory

In previous chapters, we have introduced categorical language, and see how category plays important role in algebraic topology. In Algebric topology we always focus on difference functors, trying to find some topological invariant that can distinguish different topological spaces. Homotopy theory is one of the most important branch of algebraic topology, which study topological spaces up to homotopy equivalence. In this chapter we will introduce some basic concepts of homotopy theory.

Definition 7.0.1 (Homotopy) Let X and Y be two topological spaces, two continuous maps $f, g : X \longrightarrow Y$ are said to be homotopic if there exists a continuous map $H : X \times I \longrightarrow Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$, where $I = [0, 1]$ is the unit interval. The map H is called a homotopy between f and g.

Definition 7.0.2 (Homotopy Equivalence) Let X and Y be two topological spaces, we say X and Y are homotopy equivalent if there exist continuous maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ such that $g \circ f$ is homotopic to the identity map id_X and $f \circ g$ is homotopic to the identity map id_Y. In this case, the maps f and g are called homotopy equivalences.

Example 7.0.3 Consider the unit circle $S^1 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$ and the punctured plane $\mathbb{R}^2 \backslash \{0\}$, then the map $f : S^1 \longrightarrow \mathbb{R}^2 \backslash \{0\}$ defined by $f(x, y) = (x, y)$ and the map $g : \mathbb{R}^2 \backslash \{0\} \longrightarrow S^1$ defined by $g(x, y) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$ are homotopy equivalences, since $g \circ f = \mathrm{id}_{S^1}$ and $f \circ g$ is homotopic to $\mathrm{id}_{\mathbb{R}^2 \backslash \{0\}}$ via the homotopy

$$H : (\mathbb{R}^2 \backslash \{0\}) \times I \longrightarrow \mathbb{R}^2 \backslash \{0\}, \quad H((x, y), t) = \left((1-t)x + t\frac{x}{\sqrt{x^2 + y^2}}, (1-t)y + t\frac{y}{\sqrt{x^2 + y^2}}\right)$$

Homotopy equivalence is an equivalence relation on the category of topological spaces, and we can form the homotopy category **hTop** by taking topological spaces as objects and homotopy classes of continuous maps as morphisms. Homotopy equivalence is a weaker notion than homeomorphism, since homeomorphic spaces are always homotopy equivalent, but the converse is not true in general. For example, the unit circle S^1 and the punctured plane $\mathbb{R}^2 \backslash \{0\}$ are homotopy equivalent but not homeomorphic.

Our fundamental group and fundamental groupoids are defined on the homomorpy class of one-dimension, recall that

$$\pi_1(X, x_0) = \mathrm{Hom}_{\prod(X)(x_0,x_0)} = \pi(\prod(X), x_0) \tag{7.1}$$

which we consider the foundamental group as the automorphism group of the fundamental groupoid as a category at object x_0. Geometrically, all the automorphism form the loops structures at x_0 which is S^1 up to homotopy, so what we can conclude is that $\pi_1(X, x_0)$ actually shows how we can map

S^1 into X based at x_0 up to homotopy. This is a very important observation, and actually is the geometrical definition of fundamental group.
π_1 helps us to detect whether a one-dimensional 'rubber band ' cen be contracted to a point in the space X of not. If $\pi_1(X, x_0)$ is not trivial, then there exists at least one one-dimensional hole in our space near x_0.
So what we will do next is obvious, we will generalize the fundamental group to higher dimension, that is, we want to study how we can map S^n into X based at x_0 up to homotopy for any $n \geq 1$. This leads us to the definition of higher homotopy groups, which helps us to detect higher dimensional holes in the space X near x_0 just by simply studying the structure of these groups.

Definition 7.0.4 (n-cube and its boundary) Let $I = [0, 1]$ be the unit interval. For any $n \in \mathbb{N} \setminus \{0\}$, the n-cube is defined as:

$$I^n = \underbrace{I \times I \times \cdots \times I}_{n \text{ times}} = \{(t_1, t_2, \ldots, t_n) \mid t_i \in I, \ \forall 1 \leq i \leq n\} \subset \mathbb{R}^n$$

The boundary of I^n is:

$$\partial I^n = \cup_{i=1}^n \{(t_1, t_2, \ldots, t_n) \in I^n \mid t_i \in \{0, 1\}\}$$

So the n-th homotopy group is the structures of all mapping from $n-th$ cube to the topological space X with the boundary ∂I^n mapped to the base point x_0 up to homotopy. By the identification of the boundary ∂I^n to a point, we can see that this is equivalent to study all the mapping from S^n to X based at x_0 up to homotopy, in mathematical words, $I^n/\partial I^n \cong S^n$.

Definition 7.0.5 (Higher homotopy groups) Let (X, x_0) be a pointed topological space, for any $n \in \mathbb{N} \setminus \{0\}$, the n-th homotopy group of (X, x_0) is defined as

$$\pi_n(X, x_0) = \{f : (I^n, \partial I^n) \longrightarrow (X, x_0)\}/\sim$$

where $\sim$ is the homotopy relation between continuous maps that fix the boundary ∂I^n pointwise. The group operation is given by concatenation of maps along one coordinate direction.

$$\begin{tikzcd}
(I^n, \partial I^n) \arrow[dr, "p_1"] \arrow[drr, "\pi_n(X,x_0)/\sim"] & & \\
& (I^n/\partial I^n, \partial I^n/\partial I^n) \arrow[r] \arrow[dl] & (X, x_0) \\
(S^n, s_0) \arrow[urr, "\pi_n(X,x_0)/\sim"] & &
\end{tikzcd}$$

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Supplementary Explanations

Academic Integrity Statement

My work is entirely the independent creation of me, who has thoroughly understood and mastered all content presented herein. Throughout the writing process, apart from the specifically cited references, every idea and formulation originates solely from the my own reasoning. No content has been plagiarized from the creative work or ideas of others. Furthermore, I did not use any generative AI tools to produce any portion of my work.

Explanations of Completeness

This work constitutes the midterm submission version of a year-long undergraduate thesis project and therefore exhibits a limited degree of overall completion. Specific planned additions and expansions are outlined as follows:

Chapter 2: To be supplemented with foundational category theory content, lncluding but not limited to fibrations, the Yoneda lemma, and related concepts will be used in the rest of my work.

Chapter 3: Requires a detailed exposition of fundamental groups and fundamental groupoids, more discription about definition of basic homotopy definition.

Chapter 4: More propositions and theorems need to be expanded in the categorical Galois connection, establishing the necessary groundwork of the bridge between Galois theory and covering space theory which will come later.

Chapter 5: Need a detailed and formal development of categorized Galois theory.

Chapter 6: Need a detailed and formal development of categorized covering space theory. A central focus of this chapter and also a major objective of my entire paper, will be the explicit construction of the categorical connection between Galois theory and the classification theory of covering space.

The content in Chapter 7 and beyond will be detailed discussed in the folloing section as part of the proposed future study plan.