VISUALIZING THE DISTRIBUTION OF PRIME NUMBERS

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Abstract

The prime numbers have always been at the core of number theory. Since ancient times, the mystery of the prime numbers has captivated countless mathematicians, driving them to explore their laws of primes. This project aims to visualize the distribution of prime numbers within the natural numbers, by using some graphical depictions (such as the Ulam Spiral).

Ulam Spiral

The Ulam Spiral was created by the Polish mathematician Stanislaw Ulam (1963).

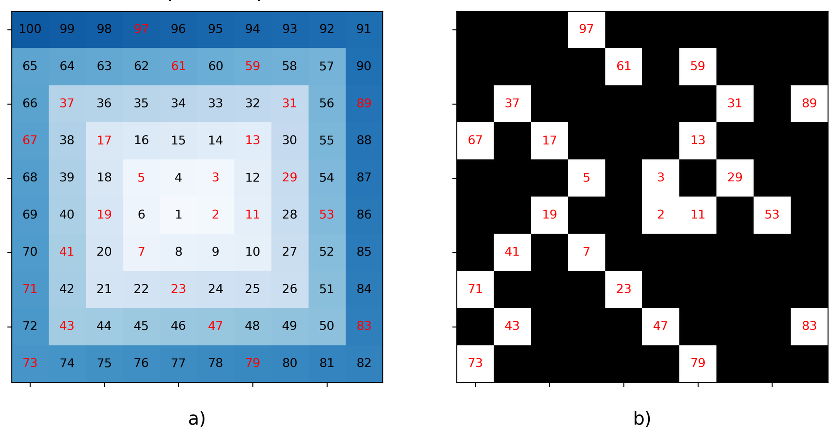


Figure 1: Ulam Spiral up to 100

The natural numbers are arranged along a counterclockwise spiral, and the prime numbers are highlighted in red. Notice that all perfect squares are placed along two semi-diagonals.

Increasing the scale of the plot allows for a more intuitive understanding of the distribution of prime numbers within a larger range. In particular, many prime numbers are observed to be gathered along certain diagonal lines.

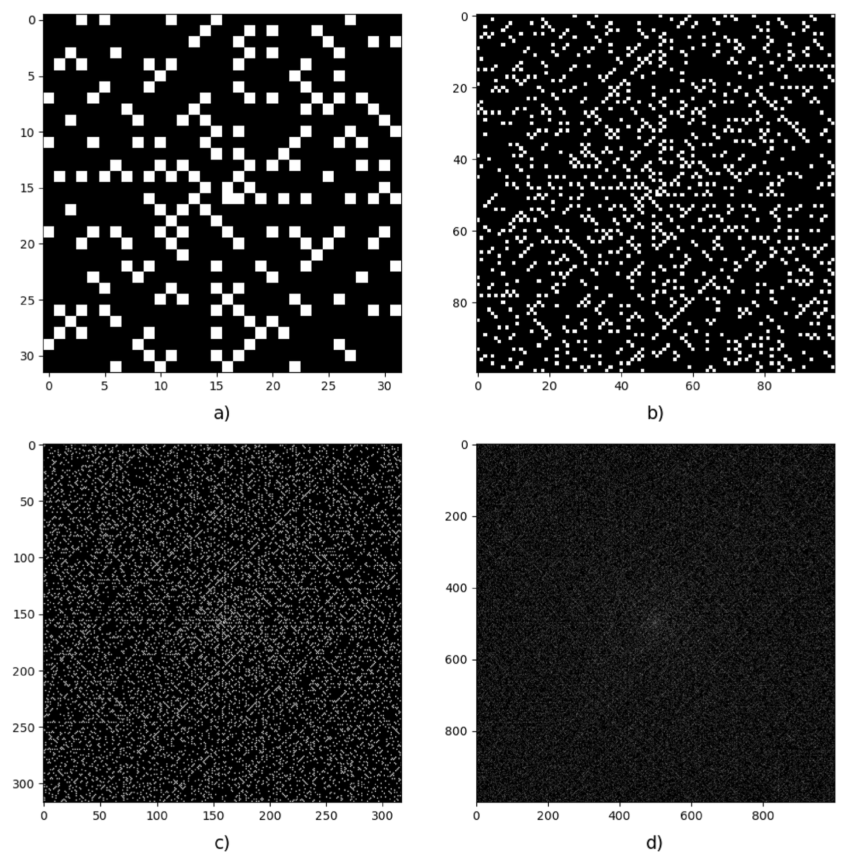


Figure 2: Ulam Spiral showing prime numbers up to **a)** 1000 **b)** 10000 **c)** 100000 **d)** 1000000

In order to observe the patterns arisen in the Ulam Spiral we may compare with the random distribution and the distribution following the asymptotic estimate for the size of the n-th prime number.

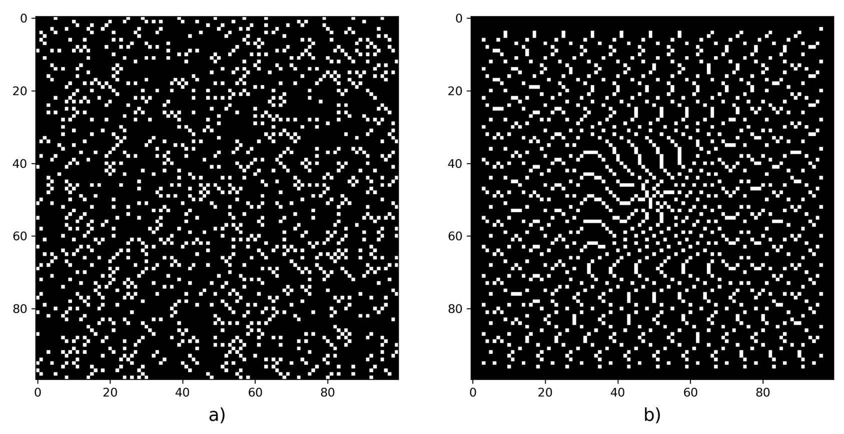


Figure 3: Comparison with random distribution and asymptotic estimate of the n-th prime: **a)** There are 1229 prime numbers up to 10000. The picture shows how these would appear if they were distributed randomly. **b)** The distribution of $\lfloor n \log n \rfloor$, $(1 \le n \le 1229)$, from the asymptotic estimate: $\lim_{n\to\infty} \frac{p_n}{n \log n} = 1$, where p_n is the n-th prime.

Klauber Triangle

The Klauber Triangle first proposed by the American herpetologist Laurence Monroe Klauber is another way to visualize the distribution of prime.

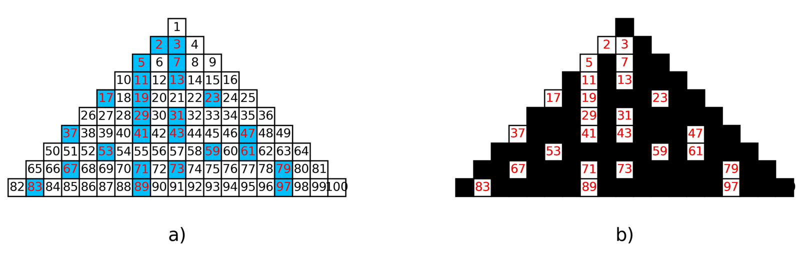


Figure 4: Klauber Triangle up to 100: All natural numbers are listed from the top to bottom, and from left to right in each row. The last integer of the n-th row is precisely n^2 because of the arithmetic formula $1+3+5+...+(2n-1)=n^2$.

Klauber Triangle (countine)

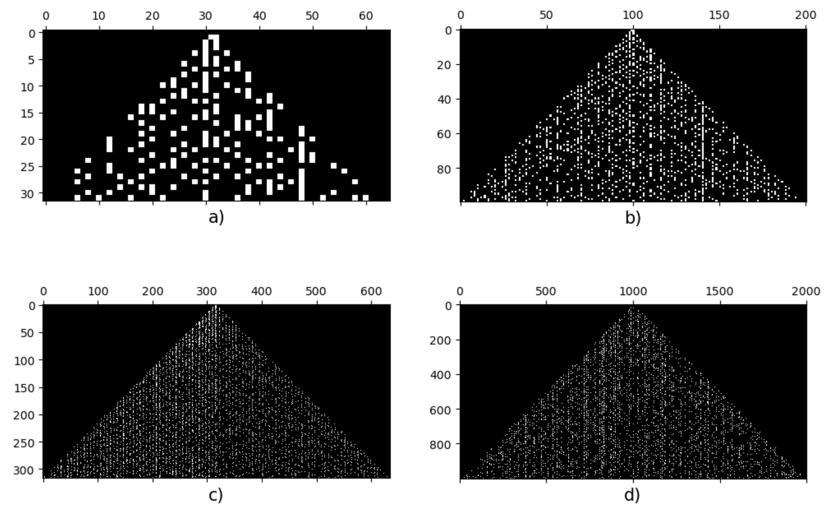


Figure 5: Klauber Triangle showing prime numbers up to **a)** 1000 **b)** 10000 **c)** 100000 **d)** 1000000

Sacks Spiral

The Sacks Spiral is a mathematical curve discovered by Daniel Sack(1994). Each natural number n is placed on the point whose angle and distance from origin are given by $\sqrt{n}$ and $2\pi\sqrt{n}$, respectively.

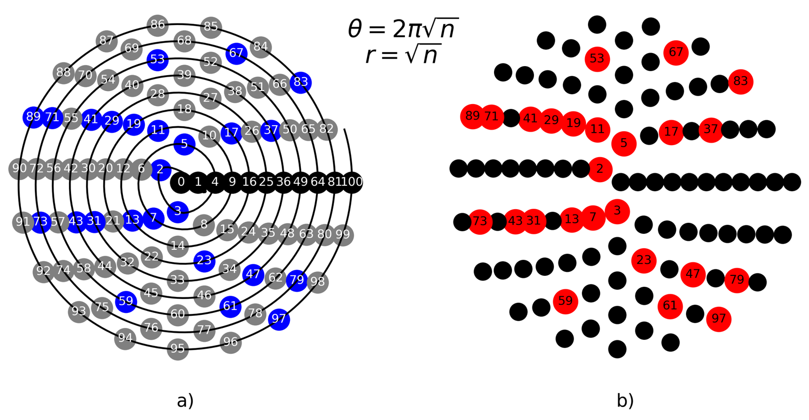


Figure 6: Sacks Spiral up to 100: In Sacks Spiral, all squares numbers are located on the positive x-axis. This is because the root of perfect square is an integer, so 2π times integer is actually a full period.

The Sacks Spiral exhibits remarkably beautiful and harmonious shapes at large scales.

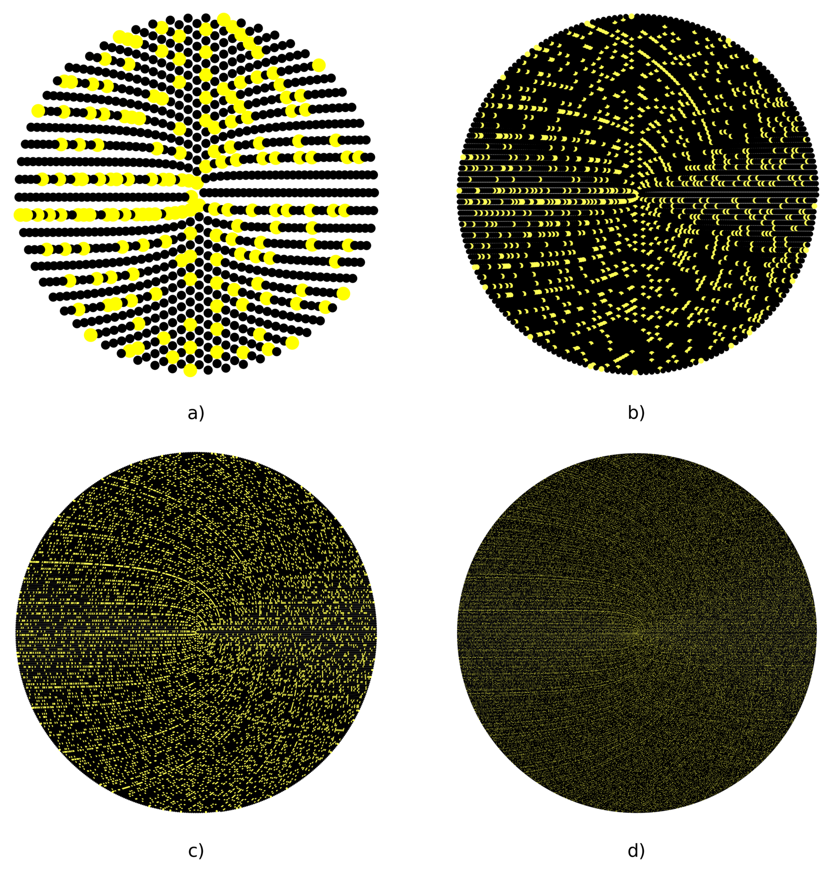
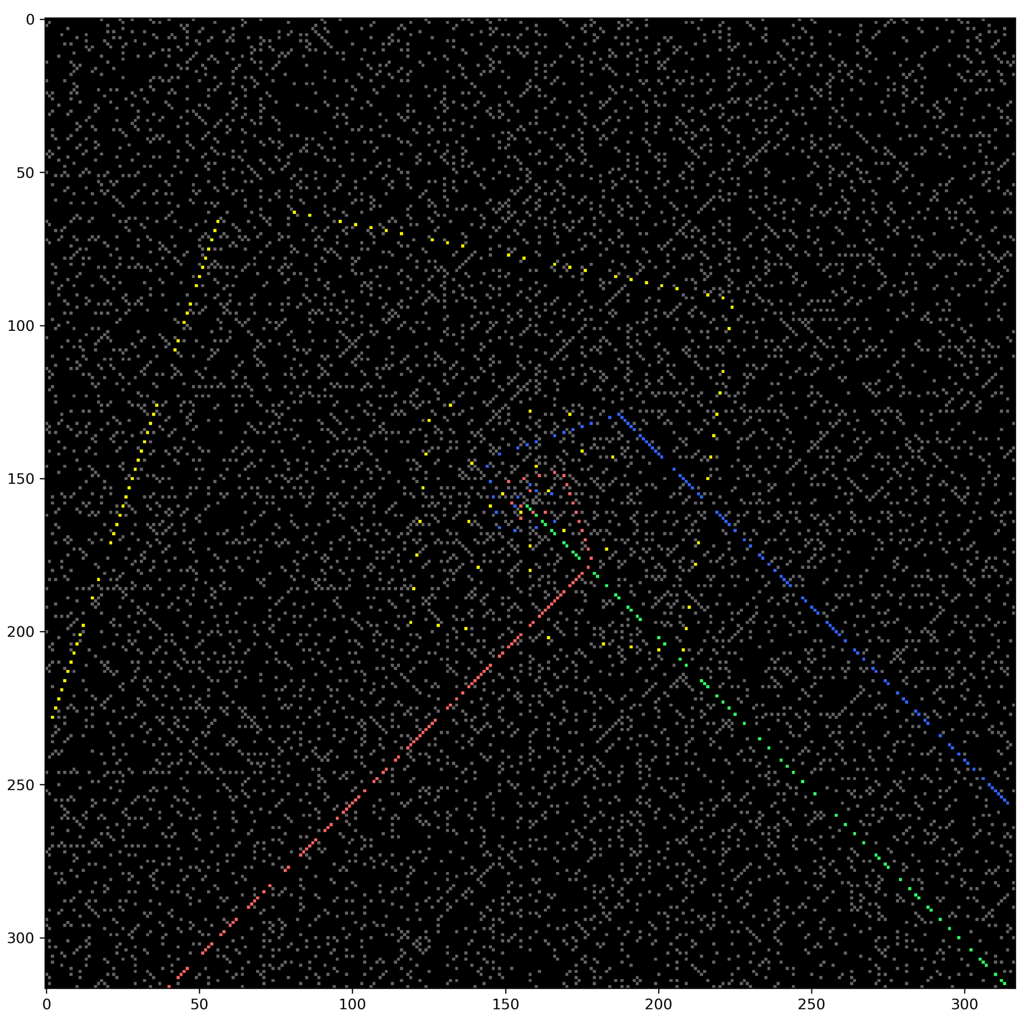


Figure 7: Sacks Spiral showing prime numbers up to **a)** 1000 **b)** 10000 **c)** 100000 **d)** 1000000

Prime-Rich Polynomials

In the Ulam Spiral we may oberve that some lines contain more prime numbers than others. We call these lines prime-rich. Actually, these segments can be discribed by polynomials.



Prime-Rich Polynomials (continued)

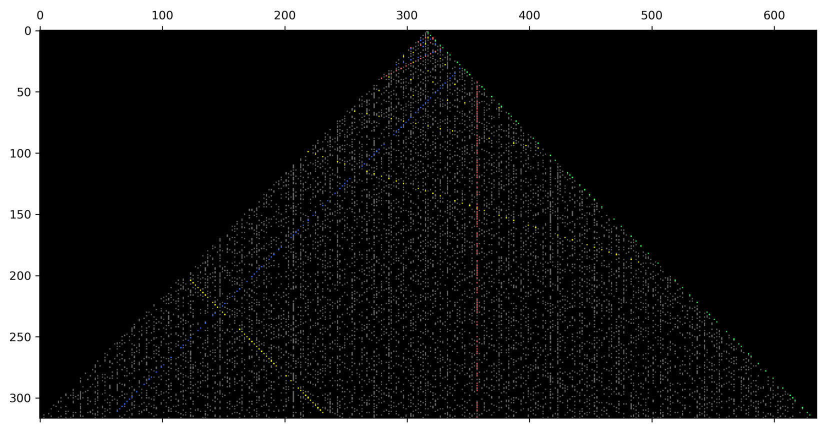


Figure 8: Ulam's and Klauber's showing prime numbers up to 100000

Examples of prime generating polynomials			
Polynomials	Number of vlaues under 100000	Number of primes under 100000	prime rate (%)
$4n^2+4n+59$	158	102	64.56
$4n^2+4n-1$	159	66	41.51
$4n^2+2n+41$	159	111	69.81
$4n^2-1260n+98827$	317	226	71.29

Notice that in the Ulam Spiral we may as well represent polynomials corresponding to lines with slope different that ±1. For instance, we consider the polynomial $4n^2-1260n+98827$, which by [1].

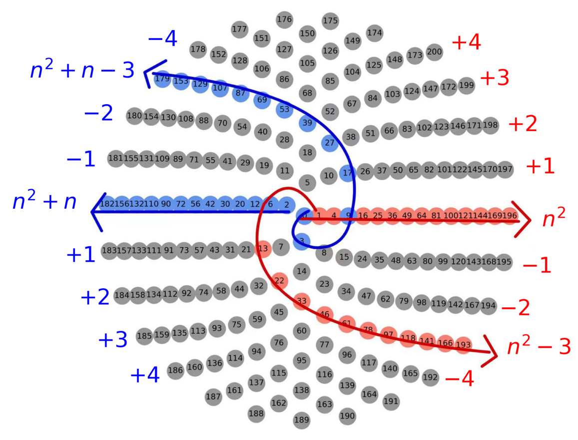


Figure 9: In the Sacks Spiral, not only the perfect numbers n^2 distributed in strict line, but also the number in the form n^2+n. In this picture, we can see all numbers in one single spiral can be discrebed in one polynomial, and this polynomial can be known be counting the distance from the two lines.

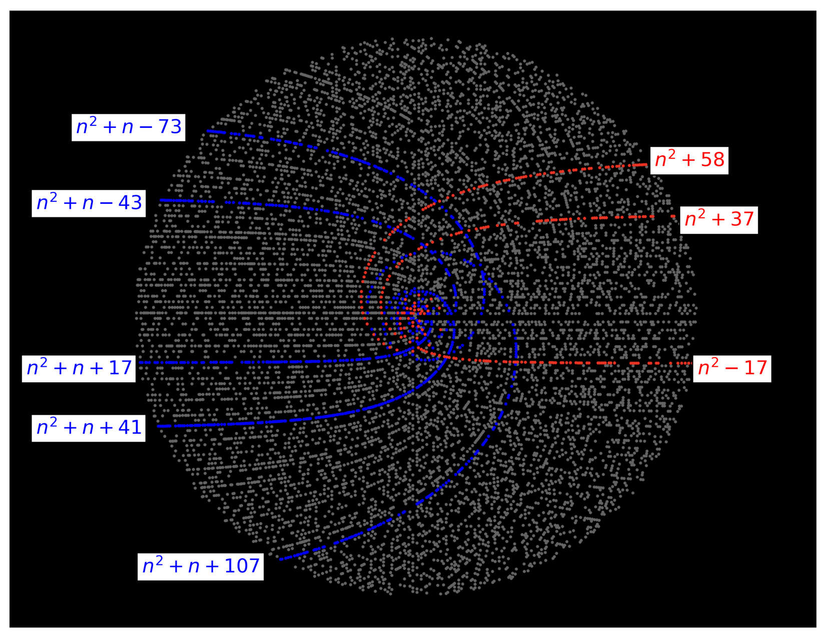


Figure 10: Sacks Spiral showing prime numbers up to 100,000

Top prime-rate polynomials: n^2+n+x $(-200 \le x \le +200)$			
Polynomials	Number of vlaues under 100000	Number of primes under 100000	prime rate (%)
n^2+n+41	316	221	69.94
$n^2+n+107$	316	177	56.01
n^2+n-73	307	160	52.12
n^2+n+17	316	145	45.89
n^2+n-43	309	136	44.01

Top prime-rate polynomials: n^2+x $(-200 \le x \le +200)$			
Polynomials $(0 \le n \le 1000)$	Number of vlaues under 100000	Number of primes under 100000	prime rate (%)
n^2+58	317	102	32.18
n^2-17	312	82	26.28
n^2+37	317	83	26.18

$p(n)=n^2+n+41$ also known as Eulers polynomial with the fact that $p(n)$ is prime for all $1 \le n \le 39$.

Conclusion

From different perspectives, visualizing the distribution of prime numbers reveals certain patterns at a large scale. Through investigating and comparing the underlying polynomials hidden within the distribution, we can gain better insights into the generation patterns and structural distribution of prime numbers.

References

[1] A. Orowski and L.J. Chmielewski. Ulam Spiral and Prime-Rich Polynomials. In L. Chmielewski et al., editors, *Computer Vision and Graphocs. Proc. ICCVG 2018, vol. 10114 of Lecture Notes in Computer Science*, pages 522-533, Warsaw, Poland, 17-19 Sep 2018. Springer.